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Memoir on the Theory of the Partitions of Numbers. Part VI: Partitions in Two-Dimensional Space, to Which is Added an Adumbration of the Theory of the Partitions in Three-Dimensional Space

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IX. *Memoir on the Theory of the Partitions of Numbers.—Part VI. Partitions in Two-dimensional Space, to which is added an Adumbration of the Theory of the Partitions in Three-dimensional Space.*

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Introduction.

I resume the subject of Part V.* of this Memoir by inquiring further into the generating function of the partitions of a number when the parts are placed at the nodes of an incomplete lattice, viz., of a lattice which is regular but made up of unequal rows. Such a lattice is the graph of the line partition of a number. In Part V. I arrived at the expression of the generating function in respect of a two-row lattice when the past magnitude is unrestricted. This was given in Art. 16 in the form

$$\text{GF}\,(\infty;\, a,\, b) = \frac{(1)+x^{b+1}(\mathbf{a}-\mathbf{b})}{(\mathbf{1})\,(\mathbf{2})\ldots(\mathbf{a}+\mathbf{1})\,.\,(\mathbf{1})\,(\mathbf{2})\ldots(\mathbf{b})}\,.$$

I remind the reader that the determination of the generating function, when the part magnitude is unrestricted, depends upon the determination of the associated lattice function (see Art. 5, *loc. cit.*). This function is assumed to be the product of an expression of known form and of another function which I termed the inner lattice function (see Art. 10, *loc. cit.*), and it is on the form of this function that the interest of the investigation in large measure depends. All that is known about it *à priori* is its numerical value when x is put equal to unity (Art. 10, *loc. cit.*). The lattice function was also exhibited as a sum of sub-lattice functions, and it was shown that the generating function, when the part magnitude is restricted, may be expressed as a linear function of them. These sub-lattice functions are intrinsically interesting, but it will be shown in what follows that they are not of vital importance to the investigation. In fact, the difficulty of constructing them has been turned by the

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formation and solution of certain functional equations which lead in the first place to the required generating functions, and in the second place to an exhibition of the forms of the sub-lattice functions. To previous definitions I here add the definition of the inner lattice function when there is a restriction upon the part magnitude, and it will be shown that the generating, lattice, and inner lattice functions satisfy certain functional equations both when there is not and when there is a restriction upon the part magnitude.

There are two methods of investigation available. We may commence with a study of the Greek-letter successions (Art. 6, *et seq.*, *loc. cit.*) from which the lattice functions are derived, and having obtained the functional equations which they satisfy, proceed thence to those satisfied by the generating and inner lattice functions; or we may reverse the process, and, by a prior determination of the equations appertaining to the generating functions, arrive at those satisfied by the lattice and inner lattice functions.

Both methods have been of service.

The results, herein achieved, are complete so far as the lattice of unequal rows and the particular question under consideration are concerned. They are elegant and algebraically interesting. In proof of this, it will suffice to say that the generating function is unaltered when the lattice is changed into its conjugate. The subject thus swarms with algebraical relations which are established intuitively.

Other results are obtained of a more general and extensive character which mark out the path of further investigation.

Art. 1. I recall that for the lattice of two unequal rows, containing a, b nodes respectively, the established results are

$$\text{Inner lattice function} = \text{IL}(\infty; a, b) = 1 + x^{b+1}\frac{(\mathbf{a-b})}{(\mathbf{1})};$$

$$\text{Lattice function} = \text{L}(\infty; a, b) = \frac{(\mathbf{1})(\mathbf{2})\dots(\mathbf{a+b})}{(\mathbf{2})(\mathbf{3})\dots(\mathbf{a+1}).(\mathbf{1})(\mathbf{2})\dots(\mathbf{b})}\left\{1 + x^{b+1}\frac{(\mathbf{a-b})}{(\mathbf{1})}\right\};$$

$$\text{Generating function} = \text{GF}(\infty; a, b) = \frac{1 + x^{b+1}\dfrac{(\mathbf{a-b})}{(\mathbf{1})}}{(\mathbf{2})(\mathbf{3})\dots(\mathbf{a+1}).(\mathbf{1})(\mathbf{2})\dots(\mathbf{b})}.$$

We have yet to determine $\text{IL}(l; a, b)$, $\text{L}(l; a, b)$, $\text{GF}(l; a, b)$, where an inner lattice function, for a restricted part magnitude, is defined by the relation

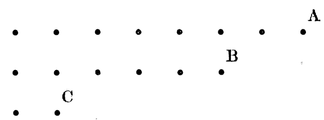
$$\text{L}(l; a_1, a_2, \dots, a_n)$$

$$= (\mathbf{1})(\mathbf{2})\dots(\Sigma\mathbf{a})\frac{(\mathbf{l+n})\dots(\mathbf{l+a_1+n-1}).(\mathbf{l+n-1})\dots\dots(\mathbf{l+a_2+n-2})\dots\dots(\mathbf{l+1})\dots(\mathbf{l+a_n})}{(\mathbf{n})\dots(\mathbf{a_1+n-1}).(\mathbf{n-1})\dots(\mathbf{a_2+n-2})\dots\dots(\mathbf{1})\dots(\mathbf{a_n})}\text{IL}(l; a_1, a_2, \dots, a_n).$$

The Functional Equations.

Art. 2. It is convenient to begin by establishing the functional equations satisfied by the generating functions.

Suppose the lattice to have three unequal rows of a, b, c nodes respectively, and let the part magnitude be restricted by the number l,



Subject to the parts being in descending order of magnitude in each row and in each column, in every partition each node is either occupied by zero (that is, is unoccupied) or by a number greater than zero and not greater than l. In certain partitions every node is occupied; such partitions may be constructed by

(i) Placing a unit at each node,
(ii) Superposing every partition enumerated by GF $(l-1\,;\ a, b, c)$.

Hence these special, full-based partitions are clearly enumerated by

$$x^{a+b+c}\,\text{GF}\,(l-1\,;\ a, b, c).$$

Similarly those partitions which are full-based upon a *contained* lattice specified by the line partition $(a'b'c')$ are enumerated by

$$x^{a'+b'+c'}\text{GF}\,(l-1\,;\ a', b', c')\,;$$

and we are led to the relation

$$\text{GF}\,(l\,;\ a, b, c) = \Sigma x^{a'+b'+c'}\text{GF}\,(l-1\,;\ a', b', c'),$$

where the summation is in regard to every lattice, specified by $(a'b'c')$, which is contained in the lattice specified by (abc).

Art. 3. If from the partitions enumerated by GF $(l\,;\ a, b, c)$ we subtract those enumerated by $x^{a+b+c}\text{GF}\,(l-1\,;\ a, b, c)$, we have remaining, in the case of three unequal rows, partitions which *include* those enumerated by each of the three generating functions

$$\text{GF}\,(l\,;\ a-1, b, c),\quad \text{GF}\,(l\,;\ a, b-1, c),\quad \text{GF}\,(l\,;\ a, b, c-1)\,;$$

and which, by a well-known principle of the combinatory analysis, are enumerated by

$$\begin{aligned}&\text{GF}\,(l\,;\ a-1, b, c)+\text{GF}\,(l\,;\ a, b-1, c)+\text{GF}\,(l\,;\ a, b, c-1)\\ &\quad -\text{GF}\,(l\,;\ a-1, b-1, c)-\text{GF}\,(l\,;\ a-1, b, c-1)-\text{GF}\,(l\,;\ a, b-1, c-1)\\ &\quad +\text{GF}\,(l\,;\ a-1, b-1, c-1).\end{aligned}$$

Hence the functional equation

$$\begin{aligned}\mathrm{GF}(l;\, a, b, c) - x^{a+b+c}\mathrm{GF}(l-1;\, a, b, c) \\ = \mathrm{GF}(l;\, a-1, b, c) + \mathrm{GF}(l;\, a, b-1, c) + \mathrm{GF}(l;\, a, b, c-1) \\ - \mathrm{GF}(l;\, a-1, b-1, c) - \mathrm{GF}(l;\, a-1, b, c-1) - \mathrm{GF}(l;\, a, b-1, c-1) \\ + \mathrm{GF}(l;\, a-1, b-1, c-1).\end{aligned}$$

In the general case of n unequal rows we have the theorem for $\mathrm{GF}(l;\, \alpha_1, \alpha_2, \ldots, \alpha_n)$; for if p_s be a symbol such that

$$p_s\mathrm{GF}(l;\, \alpha_1, \alpha_2, \ldots, \alpha_n) = \mathrm{GF}(l;\, \alpha_1, \alpha_2, \ldots, \alpha_s - 1, \ldots, \alpha_n),$$

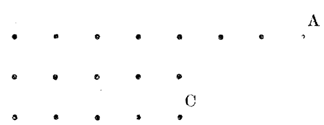
it is readily seen that

$$(1-p_1)(1-p_2)\ldots(1-p_n)\,\mathrm{GF}(l;\, \alpha_1, \alpha_2, \ldots, \alpha_n) = x^{\Sigma\alpha}\mathrm{GF}(l-1;\, \alpha_1, \alpha_2, \ldots, \alpha_n).$$

This equation is, at first sight, only true when there are no equalities between the numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$; but in the sequel, when an algebraic expression of $\mathrm{GF}(l;\, \alpha_1, \alpha_2, \ldots, \alpha_n)$ has been found, it will be seen to be true universally as an algebraical identity.

Art. 4. However, the formula *may* be modified, in the direction of simplification, when the rows are not *all* unequal.

For a given lattice we require to know how many nodes may be *singly* detached and yet leave a *contained* lattice. Thus in the three-row lattice illustrated above it is clear, the rows presenting no equalities, that we may detach singly either of the nodes lettered A, B, C; but in the case now given



a nodes
b nodes
b nodes,

it is seen that we can detach either A or C only, so that the resulting functional equation is

$$\begin{aligned}\mathrm{GF}(l;\, a, b, b) - x^{a+2b}\mathrm{GF}(l-1;\, a, b, b) \\ = \mathrm{GF}(l;\, a-1, b, b) + \mathrm{GF}(l;\, a, b, b-1) - \mathrm{GF}(l;\, a-1, b, b-1),\end{aligned}$$

which is to be compared with the equation appertaining to a lattice of two unequal rows

$$\mathrm{GF}(l;\, a, b) - x^{a+b}\mathrm{GF}(l-1;\, a, b) = \mathrm{GF}(l;\, a-1, b) + \mathrm{GF}(l;\, a, b-1) - \mathrm{GF}(l;\, a-1, b-1).$$

Similarly we derive the equations

$$\begin{aligned}\mathrm{GF}(l;\, a, a, b) - x^{2a+b}\mathrm{GF}(l-1;\, a, a, b) \\ = \mathrm{GF}(l;\, a, a-1, b) + \mathrm{GF}(l;\, a, a, b-1) - \mathrm{GF}(l;\, a, a-1, b-1),\end{aligned}$$

$$\mathrm{GF}(l;\, a, a, a) - x^{3a}\mathrm{GF}(l-1;\, a, a, a) = \mathrm{GF}(l;\, a, a, a-1);$$

and also

$$\mathrm{GF}(l\,;\,a^n)-x^{na}\mathrm{GF}(l-1\,;\,a^n)=\mathrm{GF}(l\,;\,a^{n-1},\,a-1).$$

The formation of the relation in any particular case presents no difficulty. When the lattice has k singly detachable nodes, the right-hand side of the relation involves 2^k-1 terms.

Art. 5. When the part magnitude is unrestricted, or $l=\infty$, the equations become

$$\begin{aligned}(\mathbf{a}+\mathbf{b}+\mathbf{c})\,\mathrm{GF}(\infty\,;\,a,\,b,\,c)\\ &=\mathrm{GF}(\infty\,;\,a-1,\,b,\,c)+\mathrm{GF}(\infty\,;\,a,\,b-1,\,c)+\mathrm{GF}(\infty\,;\,a,\,b,\,c-1)\\ &\quad-\mathrm{GF}(\infty\,;\,a-1,\,b-1,\,c)-\mathrm{GF}(\infty\,;\,a-1,\,b,\,c-1)-\mathrm{GF}(\infty\,;\,a,\,b-1,\,c-1)\\ &\quad+\mathrm{GF}(\infty\,;\,a-1,\,b-1,\,c-1)\end{aligned}$$

$$(\Sigma\mathbf{a})\,\mathrm{GF}(\infty\,;\,a_1,\,a_2,\,\ldots,\,a_n)=\{1-(1-p_1)(1-p_2)\ldots(1-p_n)\}\,\mathrm{GF}(\infty\,;\,a_1,\,a_2,\,\ldots,\,a_n)$$

and the modified forms are easily written down.

Art. 6. The next step is to deduce the corresponding relations between lattice functions. From the relation

$$\mathrm{GF}(\infty\,;\,a_1,\,a_2,\,\ldots,\,a_n)=\frac{\mathrm{L}(\infty\,;\,a_1,\,a_2,\,\ldots,\,a_n)}{(1)(2)\ldots(\Sigma\mathbf{a})},$$

we find

$$\mathrm{L}(\infty\,;\,a,\,b,\,c)=\Sigma x^{a'+b'+c'}\frac{(1)(2)\ldots(\mathbf{a}+\mathbf{b}+\mathbf{c})}{(1)(2)\ldots(\mathbf{a}'+\mathbf{b}'+\mathbf{c}')}\,\mathrm{L}(\infty\,;\,a',\,b',\,c')\,;$$

$$\mathrm{L}(\infty\,;a,\,a)=\mathrm{L}(\infty\,;a,\,a-1)\,;$$

$$\mathrm{L}(\infty\,;a,\,b)=\mathrm{L}(\infty\,;a-1,\,b)+\mathrm{L}(\infty\,;a,\,b-1)-(\mathbf{a}+\mathbf{b}-\mathbf{1})\,\mathrm{L}(\infty\,;a-1,\,b-1)\,;$$

$$\mathrm{L}(\infty\,;a,a,a)=\mathrm{L}(\infty\,;a,\,a,\,a-1)\,;$$

$$\mathrm{L}(\infty\,;a,\,a,\,b)=\mathrm{L}(\infty\,;a,\,a-1,\,b)+\mathrm{L}(\infty\,;a,\,a,\,b-1)-(\mathbf{2a}+\mathbf{b}-\mathbf{1})\,\mathrm{L}(\infty\,;a,\,a-1,\,b-1);$$

$$\mathrm{L}(\infty\,;a,\,b,\,b)=\mathrm{L}(\infty\,;a-1,\,b,\,b)+\mathrm{L}(\infty\,;a,\,b,\,b-1)-(\mathbf{a}+\mathbf{2b}-\mathbf{1})\,\mathrm{L}(\infty\,;a-1,\,b,\,b-1);$$

$$\begin{aligned}\mathrm{L}(\infty\,;a,\,b,\,c)&=\mathrm{L}(\infty\,;a-1,\,b,\,c)+\mathrm{L}(\infty\,;a,\,b-1,\,c)+\mathrm{L}(\infty\,;a,\,b,\,c-1)\\ &\quad-(\mathbf{a}+\mathbf{b}+\mathbf{c}-\mathbf{1})\,\{\mathrm{L}(\infty\,;a-1,\,b-1,\,c)+\mathrm{L}(\infty\,;a-1,\,b,\,c-1)\\ &\qquad+\mathrm{L}(\infty\,;a,\,b-1,\,c-1)\}\\ &\quad+(\mathbf{a}+\mathbf{b}+\mathbf{c}-\mathbf{2})(\mathbf{a}+\mathbf{b}+\mathbf{c}-\mathbf{1})\,\mathrm{L}(\infty\,;a-1,\,b-1,\,c-1).\end{aligned}$$

In the case of n unequal rows, if we write symbolically,

$$p_s\,\mathrm{L}(\infty\,;a_1,\,a_2,\,\ldots,\,a_n)=\mathrm{L}(\infty\,;a_1,\,a_2,\,\ldots,\,a_s-1,\,\ldots,\,a_n),$$

$$(\Sigma\mathbf{a})(\Sigma\mathbf{a}-\mathbf{1})\ldots(\Sigma\mathbf{a}-\mathbf{m}+\mathbf{1})=\mathrm{X}^m\,;$$

then

$$(1-\mathrm{X})\,\mathrm{L}(\infty\,;\,a_1,\,a_2,\,\ldots,\,a_n)=(1-p_1\mathrm{X})(1-p_2\mathrm{X})\ldots\ldots(1-p_n\mathrm{X})\,\mathrm{L}(\infty\,;\,a_1,\,a_2,\,\ldots,\,a_n).$$

Art. 7. Continuing, for the present, to regard the part magnitude as unrestricted, we now proceed to the equations satisfied by the inner lattice functions.

Guided by the relation

$$\mathrm{L}(\infty; a_1, a_2, \ldots, a_n) = \frac{(\mathbf{1})\ldots(\mathbf{\Sigma a})}{(\mathbf{n})\ldots(\mathbf{a_1+n-1}).(\mathbf{n-1})\ldots(\mathbf{a_2+n-2})\ldots\ldots(\mathbf{2})\ldots(\mathbf{a_{n-1}+1}).(\mathbf{1})\ldots(\mathbf{a_n})}\,\mathrm{IL}(\infty; a_1, a_2, \ldots, a_n),$$

we find

$$(\mathbf{2a})\,\mathrm{IL}(\infty; a, a) = (\mathbf{a})\,\mathrm{IL}(\infty; a, a-1);$$

$$(\mathbf{a+b})\,\mathrm{IL}(\infty; a, b) = (\mathbf{a+1})\,\mathrm{IL}(\infty; a-1, b) + (\mathbf{b})\,\mathrm{IL}(\infty; a, b-1) - (\mathbf{a+1})(\mathbf{b})\,\mathrm{IL}(\infty; a-1, b-1);$$

$$(\mathbf{3a})\,\mathrm{IL}(\infty; a, a, a) = (\mathbf{a})\,\mathrm{IL}(\infty; a, a, a-1);$$

$$(\mathbf{2a+b})\,\mathrm{IL}(\infty; a, a, b) = (\mathbf{a+1})\,\mathrm{IL}(\infty; a, a-1, b) + (\mathbf{b})\,\mathrm{IL}(\infty; a, a, b-1) - (\mathbf{a+1})(\mathbf{b})\,\mathrm{IL}(\infty; a, a-1, b-1);$$

$$(\mathbf{a+2b})\,\mathrm{IL}(\infty; a, b, b) = (\mathbf{a+2})\,\mathrm{IL}(\infty; a-1, b, b) + (\mathbf{b})\,\mathrm{IL}(\infty; a, b, b-1) - (\mathbf{a+2})(\mathbf{b})\,\mathrm{IL}(\infty; a-1, b, b-1);$$

$$\begin{aligned}(\mathbf{a+b+c})\,\mathrm{IL}(\infty; c, b, c) = {} & (\mathbf{a+2})\,\mathrm{IL}(\infty; a-1, b, c) + (\mathbf{b+1})\,\mathrm{IL}(\infty; a, b-1, c) + (\mathbf{c})\,\mathrm{IL}(\infty; a, b, c-1)\\ & - (\mathbf{a+2})(\mathbf{b+1})\,\mathrm{IL}(\infty; a-1, b-1, c) - (\mathbf{a+2})(\mathbf{c})\,\mathrm{IL}(\infty; a-1, b, c-1)\\ & - (\mathbf{b+1})(\mathbf{c})\,\mathrm{IL}(\infty; a, b-1, c-1)\\ & + (\mathbf{a+2})(\mathbf{b+1})(\mathbf{c})\,\mathrm{IL}(\infty; a-1, b-1, c-1),\end{aligned}$$

and, in general, for n unequal rows, if we write symbolically,

$$q_s\,\mathrm{IL}(\infty; a_1, a_2, \ldots, a_n) = (\mathbf{a_s+n-s})\,\mathrm{IL}(\infty; a_1, a_2, \ldots, a_s-1, \ldots, a_n),$$

$$x^{\Sigma a}\,\mathrm{IL}(\infty; a_1, a_2, \ldots, a_n) = (1-q_1)(1-q_2)\ldots(1-q_n)\,\mathrm{IL}(\infty; a_1, a_2, \ldots, a_n).$$

To these may be added

$$\mathrm{IL}(\infty; a, b, c) = \Sigma x^{a'+b'+c'}\frac{(\mathbf{3})\ldots(\mathbf{a+2}).(\mathbf{2})\ldots(\mathbf{b+1}).(\mathbf{1})\ldots(\mathbf{c})}{(\mathbf{3})\ldots(\mathbf{a'+2}).(\mathbf{2})\ldots(\mathbf{b'+1}).(\mathbf{1})\ldots(\mathbf{c'})}\,\mathrm{IL}(\infty; a', b', c'),$$

which can be readily generalized.

Art. 8. I proceed at once to find an expression for the inner lattice function.

It appears to be right to seek an expression of the function which shall show at once that the sum of the coefficients therein is that which it is otherwise known to be. Thus the result

$$\mathrm{IL}(\infty; ab) = 1 + x^{b+1}\frac{(\mathbf{a-b})}{(\mathbf{1})}$$

shows at once that the sum of the coefficients is $a-b+1$.

Since the sum of the coefficients in IL (∞ ; a, b, c) is

$$(a-b+1)(a-c+2)(b-c+1)$$

it was at first conjectured that the expression might be

$$\left\{1+x^{b+1}\frac{(\mathrm{a}-\mathrm{b})}{(1)}\right\}\left\{1+x^{c+2}\frac{(\mathrm{a}-\mathrm{c})}{(2)}\right\}\left\{1+x^{c+1}\frac{(\mathrm{b}-\mathrm{c})}{(1)}\right\},$$

but this neither satisfies the functional equation nor verifies in simple particular cases.

If, in the formula

$$\mathrm{GF}(\infty\,;a,b,c)=\frac{\mathrm{IL}(\infty\,;a,b,c)}{(3)\ldots(\mathrm{a}+2)\,.\,(2)\ldots(\mathrm{b}+1)\,.\,(1)\ldots(\mathrm{c})},$$

we put IL (∞ ; a, b, c) equal to the expression above and then put $c=0$, we obtain the known result for GF (∞ ; a, b), as may be readily seen by putting the expression in the form

$$\left\{1+x^{b+1}\frac{(\mathrm{a}-\mathrm{b})}{(1)}\right\}\left\{\frac{(\mathrm{a}+2)-x^2(\mathrm{c})}{(2)}\right\}\left\{\frac{(\mathrm{b}+1)-x(\mathrm{c})}{(1)}\right\}.$$

We are therefore justified in putting IL (∞ ; a, b, c) equal to the expression with an added term which contains the factor (c).

Write, therefore,

$$\mathrm{IL}(\infty\,;a,b,c)=\left\{1+x^{b+1}\frac{(\mathrm{a}-\mathrm{b})}{(1)}\right\}\left\{1+x^{c+2}\frac{(\mathrm{a}-\mathrm{c})}{(2)}\right\}\left\{1+x^{c+1}\frac{(\mathrm{b}-\mathrm{c})}{(1)}\right\}+(\mathrm{c})\,\mathrm{F}(\infty\,;a,b,c).$$

By working out several particular cases I was led to the conjecture

$$\mathrm{F}(\infty\,;a,b,c)=\frac{1}{(2)}\{x^{b+3}(\mathrm{a}-\mathrm{b})-x^{c+2}(\mathrm{b}-\mathrm{c})\}\,;$$

and I then found that the expression

$$\left\{1+x^{b+1}\frac{(\mathrm{a}-\mathrm{b})}{(1)}\right\}\left\{1+x^{c+2}\frac{(\mathrm{a}-\mathrm{c})}{(2)}\right\}\left\{1+x^{c+1}\frac{(\mathrm{b}-\mathrm{c})}{(1)}\right\}+\frac{(\mathrm{c})}{(2)}\{x^{b+3}(\mathrm{a}-\mathrm{b})-x^{c+2}(\mathrm{b}-\mathrm{c})\}$$

does, as a fact, satisfy the functional equation.

Art. 9. Having thus, beyond doubt, established the forms of IL (∞ ; a, b) and IL (∞ ; a, b, c), I proceed to a study of the functional equations.

In the equation

$$(\mathrm{a}+\mathrm{b})\,\mathrm{IL}(\infty\,;a,b)$$
$$=(\mathrm{a}+1)\,\mathrm{IL}(\infty\,;a-1,b)+(\mathrm{b})\,\mathrm{IL}(\infty\,;a,b-1)-(\mathrm{a}+1)(\mathrm{b})\,\mathrm{IL}(\infty\,;a-1,b-1),$$

put

$$x^{-a}\{\mathrm{IL}(\infty\,;a,b)-(a+1)\,\mathrm{IL}(\infty\,;a-1,b)\}=\mathrm{V}_1(\infty\,;a,b)\,;$$

then

$$(\mathbf{a}+\mathbf{b})\,V_1(\infty;a,b)$$
$$=(\mathbf{a}+\mathbf{1})\,V_1(\infty;a-1,b)+(\mathbf{b})\,V_1(\infty;a,b-1)-(\mathbf{a}+\mathbf{1})(\mathbf{b})\,V_1(\infty;a-1,b-1),$$

which is of the same form as the original equation.

Hence, if $\text{IL}(\infty;a,b)$ be a solution of the equation,

$$x^{-a}\{\text{IL}(\infty;a,b)-(\mathbf{a}+\mathbf{1})\,\text{IL}(\infty;a-1,b)\}$$

is also a solution.

I write

$$x^{-a}\{\text{IL}(\infty;a,b)-(\mathbf{a}+\mathbf{1})\,\text{IL}(\infty;a-1,b)\}=O_a\text{IL}(\infty;a,b),$$

exhibiting the new solution as the result of the performance of a certain operation upon the original one.

Again put, in the original equation,

$$x^{-b}\{\text{IL}(\infty;a,b)-(\mathbf{b})\,\text{IL}(\infty;a,b-1)\}=V_2(\infty;a,b)=O_b\text{IL}(\infty;a,b),$$

and we find

$$(\mathbf{a}+\mathbf{b})\,V_2(\infty;a,b)$$
$$=(\mathbf{a}+\mathbf{1})\,V_2(\infty;a-1,b)+(\mathbf{b})\,V_2(\infty;a,b-1)-(\mathbf{a}+\mathbf{1})(\mathbf{b})\,V_2(\infty;a-1,b-1);$$

so that another solution is

$$V_2(\infty;a,b)=O_b\,\text{IL}(\infty;a,b).$$

I write further

$$O_{ab}\,\text{IL}(\infty;a,b)=x^{-a-b}\{\text{IL}(\infty;a,b)-(\mathbf{a}+\mathbf{1})\,\text{IL}(\infty;a-1,b)$$
$$-(\mathbf{b})\,\text{IL}(\infty;a,b-1)+(\mathbf{a}+\mathbf{1})(\mathbf{b})\,\text{IL}(\infty;a-1,b-1)\};$$

so that, from the functional equation itself,

$$O_{ab}\,\text{IL}(\infty;a,b)=\text{IL}(\infty;a,b);$$

$$O_{ab}=1;$$

and it is easy to verify that

$$O_aO_b=O_{ab}.$$

Art. 10. Since we know one solution of the equation

$$1+x^{b+1}\frac{(\mathbf{a}-\mathbf{b})}{(\mathbf{1})},\text{ or more conveniently }(\mathbf{1})+x^{b+1}(\mathbf{a}-\mathbf{b}),$$

or better still $(\mathbf{a}+\mathbf{1})-x(\mathbf{b})$, we may at once apply the operators O_a, O_b. Operating $s-1$ times successively with O_a, I find

$$O_a^{s-1}\{(\mathbf{a}+\mathbf{1})-x(\mathbf{b})\}=(\mathbf{a}+\mathbf{1})-x^s(\mathbf{b});$$

and

$$O_b\{(\mathbf{a}+\mathbf{1})-x(\mathbf{b})\}=(\mathbf{a}+\mathbf{1})-(\mathbf{b}),$$

$$O_b^{s+1}\{(\mathbf{a}+\mathbf{1})-x(\mathbf{b})\}=(\mathbf{a}+\mathbf{1})-x^{-s}(\mathbf{b}).$$

We may therefore take $(\mathbf{a}+1)$ and $(\mathbf{b})$ as the two fundamental solutions, and clearly we may always multiply a solution by any function of x which does not involve a or b.

The final expression of IL (∞ ; a, b) which I adopt is

$$\text{IL}(\infty; a, b) = \begin{vmatrix} (\mathbf{a}+1), & (\mathbf{b}) \\ x \quad , & 1 \end{vmatrix} \div \begin{vmatrix} 1, & 1 \\ x, & 1 \end{vmatrix};$$

and we will find that, expressed thus as the quotient of two determinants, it is generalizable. I might now, knowing *à posteriori* the expression for IL (∞ ; abc), proceed in a simpler manner than what follows; but I think it better to put before the reader the actual course that the investigation took.

Art. 11. In the functional equation satisfied by IL (∞ ; a, b, c), which may be written

$$\begin{aligned} &\text{IL}(\infty; a, b, c)-(\mathbf{b}+1)\,\text{IL}(\infty; a, b-1, c)-(\mathbf{c})\,\text{IL}(\infty; a, b, c-1) \\ &\qquad\qquad +(\mathbf{b}+1)(\mathbf{c})\,\text{IL}(\infty; a, b-1, c-1) \\ &\quad = x^{a+b+c}\,\text{IL}(\infty; a, b, c)+(\mathbf{a}+2)\{\text{IL}(\infty; a-1, b, c)-(\mathbf{b}+1)\,\text{IL}(\infty; a-1, b-1, c) \\ &\qquad\qquad -(\mathbf{c})\,\text{IL}(\infty; a-1, b, c-1)+(\mathbf{b}+1)(\mathbf{c})\,\text{IL}(\infty; a-1, b-1, c-1)\}, \end{aligned}$$

I write

$$\begin{aligned} \text{V}_1(\infty; a, b, c) = x^{-b-c}\{&\text{IL}(\infty; a, b, c)-(\mathbf{b}+1)\,\text{IL}(\infty; a, b-1, c) \\ &-(\mathbf{c})\,\text{IL}(\infty; a, b, c-1)+(\mathbf{b}+1)(\mathbf{c})\,\text{IL}(\infty; a, b-1, c-1\}; \end{aligned}$$

and thence derive the relation

$$\begin{aligned} &(\mathbf{a}+\mathbf{b}+\mathbf{c})\,\text{V}_1(\infty; a, b, c) \\ &\quad = (\mathbf{a}+2)\,\text{V}_1(\infty; a-1, b, c)+(\mathbf{b}+1)\,\text{V}_1(\infty; a, b-1, c)+(\mathbf{c})\,\text{V}_1(\infty; a, b, c-1) \\ &\qquad -(\mathbf{a}+2)(\mathbf{b}+1)\,\text{V}_1(\infty; a-1, b-1, c)-(\mathbf{a}+2)(\mathbf{c})\,\text{V}_1(\infty; a-1, b, c-1) \\ &\qquad -(\mathbf{b}+1)(\mathbf{c})\,\text{V}_1(\infty; a, b-1, c-1)+(\mathbf{a}+2)(\mathbf{b}+1)(\mathbf{c})\,\text{V}_1(\infty; a-1, b-1, c-1). \end{aligned}$$

Comparing this with the functional equation it is clear that V_1 (∞ ; a, b, c), as defined, is a solution.

Proceeding similarly we find six solutions which I exhibit as operations performed upon IL (∞ ; a, b, c) as follows :—

$$x^{-a}\{\text{IL}(\infty; a, b, c)-(\mathbf{a}+2)\,\text{IL}(\infty; a-1, b, c)\} = \text{O}_a\text{IL}(\infty; a, b, c);$$

$$x^{-b}\{\text{IL}(\infty; a, b, c)-(\mathbf{b}+1)\,\text{IL}(\infty; a, b-1, c)\} = \text{O}_b\text{IL}(\infty; a, b, c);$$

$$x^{-c}\{\text{IL}(\infty; a, b, c)-\ (\mathbf{c})\ \ \text{IL}(\infty; a, b, c-1)\} = \text{O}_c\text{IL}(\infty; a, b, c);$$

$$x^{-b-c}\{\text{IL}(\infty;a,b,c)-(\mathbf{b+1})\,\text{IL}(\infty;a,b-1,c)-(\mathbf{c})\,\text{IL}(\infty;a,b,c-1)$$
$$+(\mathbf{b+1})(\mathbf{c})\,\text{IL}(\infty;a,b-1,c-1)\}=\text{O}_{bc}\text{IL}(\infty;a,b,c);$$

$$x^{-c-a}\{\text{IL}(\infty;a,b,c)-(\mathbf{c})\,\text{IL}(\infty;a,b,c-1)-(\mathbf{a+2})\,\text{IL}(\infty;a-1,b,c)$$
$$+(\mathbf{c})(\mathbf{a+2})\,\text{IL}(\infty;a-1,b,c-1)\}=\text{O}_{ca}\text{IL}(\infty;a,b,c);$$

$$x^{-a-b}\{\text{IL}(\infty;a,b,c)-(\mathbf{a+2})\,\text{IL}(\infty;a-1,b,c)-(\mathbf{b+1})\,\text{IL}(\infty;a,b-1,c)$$
$$+(\mathbf{a+2})(\mathbf{b+1})\,\text{IL}(\infty;a-1,b-1,c)\}=\text{O}_{ab}\text{IL}(\infty;a,b,c).$$

I further write

$$\text{O}_{abc}\text{IL}(\infty;a,b,c)$$
$$=x^{-a-b-c}\{\text{IL}(\infty;a,b,c)-(\mathbf{a+2})\,\text{IL}(\infty;a-1,b,c)-(\mathbf{b+1})\,\text{IL}(\infty;a,b-1,c)$$
$$-(\mathbf{c})\,\text{IL}(\infty;a,b,c-1)+(\mathbf{a+2})(\mathbf{b+1})\,\text{IL}(\infty;a-1,b-1,c)$$
$$+(\mathbf{a+2})(\mathbf{c})\,\text{IL}(\infty;a-1,b,c-1)+(\mathbf{b+1})(\mathbf{c})\,\text{IL}(\infty;a,b-1,c-1)$$
$$-(\mathbf{a+2})(\mathbf{b+1})(\mathbf{c})\,\text{IL}(\infty;a-1,b-1,c-1)\};$$

and it is easy to establish the operator relations

$$\text{O}_a\text{O}_b\text{O}_c=1,$$

$$\text{O}_a\text{O}_b=\text{O}_{ab},\quad \text{O}_a\text{O}_c=\text{O}_{ac},\quad \text{O}_b\text{O}_c=\text{O}_{bc},$$

$$\text{O}_a\text{O}_{bc}=\text{O}_b\text{O}_{ac}=\text{O}_c\text{O}_{ab}=\text{O}_{abc}=1.$$

Art. 12. I now operate with these operators upon the known solution of the functional equation. To clear it of fractions I multiply throughout by $(1)^2(2)$. Operating m times in succession with O_c I obtain the result

$$(\mathbf{1})-x^{a+1}(\mathbf{2})(\mathbf{b+1})-x^{a+2}(\mathbf{a+1})+x^{b+2}(\mathbf{b})+x^{a+2b+4}(\mathbf{a-b})$$
$$-x^{m+1}\{(\mathbf{2})+x^{b+2}(1-x)(\mathbf{a-b-1})-x^{2b+3}(\mathbf{2a-2b})\}(\mathbf{c})$$
$$+x^{-2m+3}\{(\mathbf{1})+x^{b+2}(\mathbf{a-b})\}(\mathbf{c-1})(\mathbf{c}).$$

Whence I conclude that

$$\text{P}_1=(\mathbf{1})-x^{a+1}(\mathbf{2})(\mathbf{b+1})-x^{a+2}(\mathbf{a+1})+x^{b+2}(\mathbf{b})+x^{a+2b+4}(\mathbf{a-b}),$$

$$\text{P}_2=\{(\mathbf{2})+x^{b+2}(1+x)(\mathbf{a-b-1})-x^{2b+3}(\mathbf{2a-2b})\}(\mathbf{c}),$$

$$\text{P}_3=\{(\mathbf{1})+x^{b+2}(\mathbf{a-b})\}(\mathbf{c-1})(\mathbf{c}),$$

are solutions of the functional equation.

I find that

$$\text{O}_c\text{P}_1=\text{P}_1,\quad \text{O}_c\text{P}_2=x^{-1}\text{P}_2,\quad \text{O}_c\text{P}_3=x^{-2}\text{P}_3;$$

but new solutions are obtained by operating upon P_1, P_2, and P_3 with O_a and O_b.

Operating with O_b, m times successively, upon P_3 I obtain

$$-x\,(\mathbf{c}-\mathbf{1})\,(\mathbf{c})\,\{(\mathbf{b}+\mathbf{1})-x^{m-1}\,(\mathbf{a}+\mathbf{2})\},$$

and I draw the inference that

$$(\mathbf{c}-\mathbf{1})\,(\mathbf{c})\,(\mathbf{b}+\mathbf{1}) \quad \text{and} \quad (\mathbf{c}-\mathbf{1})\,(\mathbf{c})\,(\mathbf{a}+\mathbf{2})$$

are solutions.

Further, operating with O_b, m times successively, upon P_2, I obtain

$$(\mathbf{a}+\mathbf{1})\,(\mathbf{a}+\mathbf{2})\,(\mathbf{c})\,x^{m}-(\mathbf{b})\,(\mathbf{b}+\mathbf{1})\,(\mathbf{c})\,x^{-m+2}\,;$$

and it thence appears that

$$(\mathbf{a}+\mathbf{1})\,(\mathbf{a}+\mathbf{2})\,(\mathbf{c}) \quad \text{and} \quad (\mathbf{b})\,(\mathbf{b}+\mathbf{1})\,(\mathbf{c})$$

are solutions.

Again, operating with O_b, m times successively, upon P_1, I obtain

$$(\mathbf{a}+\mathbf{1})\,(\mathbf{a}+\mathbf{2})\,(\mathbf{b}+\mathbf{1})-(\mathbf{b})\,(\mathbf{b}+\mathbf{1})\,(\mathbf{a}+\mathbf{2})\,x^{-m+1}\,;$$

and the conclusion is that

$$(\mathbf{a}+\mathbf{1})\,(\mathbf{a}+\mathbf{2})\,(\mathbf{b}+\mathbf{1}) \quad \text{and} \quad (\mathbf{b})\,(\mathbf{b}+\mathbf{1})\,(\mathbf{a}+\mathbf{2})$$

are solutions.

No other fundamental solutions are obtainable by operating with O_a, O_b, and O_c upon P_1, P_2, and P_3, and clearly we have no need to consider the other operators because of the relations between them.

We have thus six fundamental solutions

$$(\mathbf{a}+\mathbf{1})\,(\mathbf{a}+\mathbf{2})\,(\mathbf{b}+\mathbf{1}), \quad (\mathbf{a}+\mathbf{1})\,(\mathbf{a}+\mathbf{2})\,(\mathbf{c}), \quad (\mathbf{b})\,(\mathbf{b}+\mathbf{1})\,(\mathbf{c}),$$

$$(\mathbf{b})\,(\mathbf{b}+\mathbf{1})\,(\mathbf{a}+\mathbf{2}), \quad (\mathbf{c}-\mathbf{1})\,(\mathbf{c})\,(\mathbf{a}+\mathbf{2}), \quad (\mathbf{c}-\mathbf{1})\,(\mathbf{c})\,(\mathbf{b}+\mathbf{1}).$$

Art. 13. The known solution of the functional equation from which these solutions have been derived can now be expressed in terms of these. Since it has been found that

$$O_c^{\,m}\,(\mathbf{1})^2\,(\mathbf{2})\,\mathrm{IL}\,(\,\infty\,;\,a,\,b,\,c) = P_1-x^{m+1}P_2+x^{-2m+3}P_3,$$

we have

$$(\mathbf{1})^2\,(\mathbf{2})\,\mathrm{IL}\,(\,\infty\,;\,a,\,b,\,c) = P_1-xP_2+x^3P_3\,;$$

and, putting $m = 0$ in results obtained above, it appears that

$$P_1 = (\mathbf{a}+\mathbf{1})\,(\mathbf{a}+\mathbf{2})\,(\mathbf{b}+\mathbf{1})-x\,(\mathbf{b})\,(\mathbf{b}+\mathbf{1})\,(\mathbf{a}+\mathbf{2}),$$

$$P_2 = (\mathbf{a}+\mathbf{1})\,(\mathbf{a}+\mathbf{2})\,(\mathbf{c})-x^2\,(\mathbf{b})\,(\mathbf{b}+\mathbf{1})\,(\mathbf{c}),$$

$$P_3 = -x\,(\mathbf{c}-\mathbf{1})\,(\mathbf{c})\,(\mathbf{b}+\mathbf{1})+(\mathbf{c}-\mathbf{1})\,(\mathbf{c})\,(\mathbf{a}+\mathbf{2}).$$

whence

$$(1)^2(2)\,\mathrm{IL}(\infty; a, b, c) = (\mathbf{a}+1)(\mathbf{a}+2)(\mathbf{b}+1) - x(\mathbf{b})(\mathbf{b}+1)(\mathbf{a}+2) - x(\mathbf{a}+1)(\mathbf{a}+2)(\mathbf{c}) + x^3(\mathbf{b})(\mathbf{b}+1)(\mathbf{c}) + x^3(\mathbf{c}-1)(\mathbf{c})(\mathbf{a}+2) - x^4(\mathbf{c}-1)(\mathbf{c})(\mathbf{b}+1),$$

or

$$\mathrm{IL}(\infty; a, b, c) = \begin{vmatrix} (\mathbf{a}+1)(\mathbf{a}+2), & (\mathbf{b})(\mathbf{b}+1), & x(\mathbf{c}-1)(\mathbf{c}) \\ x(\mathbf{a}+2) \quad , & (\mathbf{b}+1) \quad , & (\mathbf{c}) \\ x^3 \quad , & x \quad , & 1 \end{vmatrix} \div \begin{vmatrix} 1, & 1, & x \\ x, & 1, & 1 \\ x^3, & x, & 1 \end{vmatrix},$$

a satisfactory representation of the inner lattice function.

Art. 14. Passing now to the consideration of the inner lattice function of the order 4, viz., $\mathrm{IL}(\infty; a, b, c, d)$, and guided by the above results, I put

$$\begin{array}{lll} A_1 = (\mathbf{a}+3), & A_2 = (\mathbf{a}+2)(\mathbf{a}+3), & A_3 = (\mathbf{a}+1)(\mathbf{a}+2)(\mathbf{a}+3), \\ B_1 = (\mathbf{b}+2), & B_2 = (\mathbf{b}+1)(\mathbf{b}+2), & B_3 = (\mathbf{b})(\mathbf{b}+1)(\mathbf{b}+2)\ , \\ C_1 = (\mathbf{c}+1), & C_2 = (\mathbf{c})(\mathbf{c}+1)\ , & C_3 = (\mathbf{c}-1)(\mathbf{c})(\mathbf{c}+1)\ , \\ D_1 = (\mathbf{d})\ , & D_2 = (\mathbf{d}-1)(\mathbf{d})\ , & D_3 = (\mathbf{d}-1)(\mathbf{d}-1)(\mathbf{d})\ , \end{array}$$

and I consider the twenty-four products

$$\begin{array}{llll} A_3B_2C_1, & A_3B_2D_1, & A_3D_2C_1, & D_3B_2C_1, \\ A_3C_2B_1, & A_3D_2B_1, & A_3C_2D_1, & D_3C_2B_1, \\ B_3A_2C_1, & B_3A_2D_1, & D_3A_2C_1, & B_3D_2C_1, \\ B_3C_2A_1, & B_3D_2A_1, & D_3C_2A_1, & B_3C_2D_1, \\ C_3A_2B_1, & D_3A_2B_1, & C_3A_2D_1, & C_3D_2B_1, \\ C_3B_2A_1, & D_3B_2A_1, & C_3D_2A_1, & C_3B_2D_1, \end{array}$$

which, to suffices 3, 2, 1 in descending order, involve every permutation of the letters ABCD, three at a time.

Art. 15. I shall show that each of these products is a solution of the functional equation

$$x^{\Sigma a}\,\mathrm{IL}(\infty; a, b, c, d) = (1-q_1)(1-q_2)(1-q_3)(1-q_4)\,\mathrm{IL}(\infty; a, b, c, d);$$

for, looking at the definition of the symbol q, it is clear that

$$q_1A_3B_2C_1 = (\mathbf{a})A_3B_2C_1,\ q_2A_3B_2C_1 = (\mathbf{b})A_3B_2C_1,\ q_3A_3B_2C_1 = (\mathbf{c})A_3B_2C_1,\ q_4A_3B_2C_1 = (\mathbf{d})A_3B_2C_1,$$
$$q_1q_2A_3B_2C_1 = (\mathbf{a})(\mathbf{b})A_3B_2C_1,\ q_1q_2q_3A_3B_2C_1 = (\mathbf{a})(\mathbf{b})(\mathbf{c})A_3B_2C_1,\ \&c.$$

Hence

$$(1-q_1)(1-q_2)(1-q_3)(1-q_4)A_3B_2C_1 = \{1-(\mathbf{a})\}\{1-(\mathbf{b})\}\{1-(\mathbf{c})\}\{1-(\mathbf{d})\}A_3B_2C_1 = x^{\Sigma a}A_3B_2C_1,$$

establishing that $A_3B_2C_1$ is a solution.

In general, put

$$A_0 = B_0 = C_0 = D_0 = 1,$$

and consider the product $A_\alpha B_\beta C_\gamma D_\delta$ where A, B, C, D are in fixed alphabetical order and α, β, γ, δ is some permutation of 3, 2, 1, 0.

We find

$$q_1 A_\alpha B_\beta C_\gamma D_\delta = (\mathbf{a}-\boldsymbol{\alpha}+3)\, A_\alpha B_\beta C_\gamma D_\delta,$$

and the effect of the symbols q_2, q_3, q_4 is to multiply by $(\mathbf{b}-\boldsymbol{\beta}+2)$, $(\mathbf{c}-\boldsymbol{\gamma}+1)$, $(\mathbf{d}-\boldsymbol{\delta})$ respectively. Hence

$$(1-q_1)(1-q_2)(1-q_3)(1-q_4)\, A_\alpha B_\beta C_\gamma D_\delta$$

$$= \{1-(\mathbf{a}-\boldsymbol{\alpha}+3)\}\{1-(\mathbf{b}-\boldsymbol{\beta}+2)\}\{1-(\mathbf{c}-\boldsymbol{\gamma}+1)\}\{1-(\mathbf{d}-\boldsymbol{\delta})\}\, A_\alpha B_\beta C_\gamma D_\delta,$$

$$= x^{a-\alpha+3+b-\beta+2+c-\gamma+1+d-\delta}\, A_\alpha B_\beta C_\gamma D_\delta,$$

$$= x^{a+b+c+d}\, A_\alpha B_\beta C_\gamma D_\delta,$$

since

$$\alpha+\beta+\gamma+\delta = 1+2+3.$$

It is thus established that each of the products in question is a solution of the functional equation.

Art. 16. Hence the determinant, which is a linear function of these products, viz. :—

$$\begin{vmatrix} (\mathbf{a}+1)(\mathbf{a}+2)(\mathbf{a}+3), & (\mathbf{b})(\mathbf{b}+1)(\mathbf{b}+2), & x(\mathbf{c}-1)(\mathbf{c})(\mathbf{c}+1), & x^3(\mathbf{d}-2)(\mathbf{d}-1)(\mathbf{d}) \\ x(\mathbf{a}+2)(\mathbf{a}+3) \quad , & (\mathbf{b}+1)(\mathbf{b}+2) \quad , & (\mathbf{c})(\mathbf{c}+1) \quad , & x(\mathbf{d}-1)(\mathbf{d}) \\ x^3(\mathbf{a}+3) \quad , & x(\mathbf{b}+2) \quad , & (\mathbf{c}+1) \quad , & (\mathbf{d}) \\ x^6 \quad , & x^3 \quad , & x \quad , & 1 \end{vmatrix};$$

and I shall show that this determinant, divided by the determinant

$$\begin{vmatrix} 1, & 1, & x, & x^3 \\ x, & 1, & 1, & x \\ x^3, & x, & 1, & 1 \\ x^6, & x^3, & x, & 1 \end{vmatrix},$$

is the actual expression of IL (∞ ; a, b, c, d).

Art. 17. I first take the test of the sum of the coefficients which we know otherwise to be

$$\tfrac{1}{12}(a-b+1)(a-c+2)(a-d+3)(b-c+1)(b-d+2)(c-d+1).$$

The denominator determinant has the value $(\mathbf{1})^3(\mathbf{2})^2(\mathbf{3})$; dividing numerator and denominator by $(1)^6$, and then putting x equal to unity, we find

$$\frac{1}{12}\begin{vmatrix} (a+1)(a+2)(a+3), & b(b+1)(b+2), & (c-1)c(c+1), & (d-2)(d-1)(d) \\ (a+2)(a+3) \quad , & (b+1)(b+2)\;, & c(c+1) \quad , & (d-1)d \\ a+3 \quad , & b+2 \quad , & c+1 \quad , & d \\ 1 \quad , & 1 \quad , & 1 \quad , & 1 \end{vmatrix},$$

and herein putting $a-b+1$, $a-c+2$, $a-d+3$, $b-c+1$, $b-d+2$, $c-d+1$ separately equal to zero, we in each case find two columns becoming identical and the determinant vanishing. Hence the sum of the coefficients has the proper numerical value.

Art. 18. As a second test I will show that the quotient of determinants becomes unity on putting $a=b=c=d$.

The numerator determinant becomes

$$\begin{vmatrix} (\mathbf{a}+\mathbf{1})(\mathbf{a}+\mathbf{2})(\mathbf{a}+\mathbf{3}), & (\mathbf{a})(\mathbf{a}+\mathbf{1})(\mathbf{a}+\mathbf{2}), & x(\mathbf{a}-\mathbf{1})(\mathbf{a})(\mathbf{a}+\mathbf{1}), & x^3(\mathbf{a}-\mathbf{2})(\mathbf{a}-\mathbf{1})(\mathbf{a}) \\ x(\mathbf{a}+\mathbf{2})(\mathbf{a}+\mathbf{3}) \quad , & (\mathbf{a}+\mathbf{1})(\mathbf{a}+\mathbf{2}) \quad , & (\mathbf{a})(\mathbf{a}+\mathbf{1}) \quad , & x(\mathbf{a}-\mathbf{1})(\mathbf{a}) \\ x^3(\mathbf{a}+\mathbf{3}) \quad , & x(\mathbf{a}+\mathbf{2}) \quad , & (\mathbf{a}+\mathbf{1}) \quad , & (\mathbf{a}) \\ x^6 \quad , & x^3 \quad , & x \quad , & 1 \end{vmatrix}.$$

Transform this by taking

For New First Row—

$$1^{\text{st}}\text{ Row}+x^a(1+x+x^2)\times 2^{\text{nd}}\text{ Row}+x^{2a+1}(1+x+x^2)\times 3^{\text{rd}}\text{ Row}+x^{3a+3}\times 4^{\text{th}}\text{ Row},$$

For New Second Row—

$$2^{\text{nd}}\text{ Row}+x^a(1+x)\times 3^{\text{rd}}\text{ Row}+x^{2a+1}\times 4^{\text{th}}\text{ Row},$$

For New Third Row—

$$3^{\text{rd}}\text{ Row}+x^a\times 4^{\text{th}}\text{ Row},$$

and it becomes

$$\begin{vmatrix} 1, & 1, & x, & x^3 \\ x, & 1, & 1, & x \\ x^3, & x, & 1, & 1 \\ x^6, & x^3, & x, & 1 \end{vmatrix},$$

and thus the quotient of determinants is unity.

This verifies numerous particular cases.

Art. 19. A third test is to show that the quotient of determinants has the value

$$1+x^a+x^{2a}+x^{3a}$$

when $b=c=a$, $d=a-1$.

The proof is too long to find a place here.

All of the processes employed above are obviously valid when applied to the functional equation of order n and lead to the expression of $\mathrm{IL}(\infty; \alpha_1, \alpha_2, \ldots, \alpha_n)$ as a quotient of determinants.

Art. 20. Before proceeding further I collect together the chief results obtained above.

$$\mathrm{IL}(\infty; a, b) = \begin{vmatrix} (\mathbf{a}+1), & (\mathbf{b}) \\ x, & 1 \end{vmatrix} \div \begin{vmatrix} 1, & 1 \\ x, & 1 \end{vmatrix};$$

$$\mathrm{L}(\infty; a, b) = \frac{(1)(2)\ldots(\mathbf{a}+\mathbf{b}) \begin{vmatrix} (\mathbf{a}+1), & (\mathbf{b}) \\ x, & 1 \end{vmatrix} \div \begin{vmatrix} 1, & 1 \\ x, & 1 \end{vmatrix}}{(2)(3)\ldots(a+1).(1)(2)\ldots(b)};$$

$$\mathrm{GF}(\infty; a, b) = \frac{\begin{vmatrix} (\mathbf{a}+1), & \mathbf{b} \\ x, & 1 \end{vmatrix} \div \begin{vmatrix} 1, & 1 \\ x, & 1 \end{vmatrix}}{(2)(3)\ldots(\mathbf{a}+1).(1)(2)\ldots(\mathbf{b})};$$

$$\mathrm{IL}(\infty; a, b, c) = \begin{vmatrix} (\mathbf{a}+1)(\mathbf{a}+2), & (\mathbf{b})(\mathbf{b}+1), & x(\mathbf{c}-1)(\mathbf{c}) \\ x(\mathbf{a}+2), & (\mathbf{b}+1), & (\mathbf{c}) \\ x^3, & x, & 1 \end{vmatrix} \div \begin{vmatrix} 1, & 1, & 1 \\ x, & 1, & 1 \\ x^3, & x, & 1 \end{vmatrix};$$

$$\mathrm{L}(\infty; a, b, c) = \frac{\begin{vmatrix} (\mathbf{a}+1)(\mathbf{a}+2), & (\mathbf{b})(\mathbf{b}+1), & x(\mathbf{c}-1)(\mathbf{c}) \\ x(\mathbf{a}+2), & (\mathbf{b}+1), & (\mathbf{c}) \\ x^3, & x, & 1 \end{vmatrix} \div \begin{vmatrix} 1, & 1, & 1 \\ x, & 1, & 1 \\ x^3, & x, & 1 \end{vmatrix}}{(3)(4)\ldots(\mathbf{a}+2).(2)(3)\ldots(\mathbf{b}+1).(1)(2)\ldots(\mathbf{c})}(1)(2)\ldots(\mathbf{a}+\mathbf{b}+\mathbf{c});$$

and, not putting the denominator determinant in evidence,

$$\mathrm{GF}(\infty; a, b, c) = \frac{\begin{vmatrix} (\mathbf{a}+1)(\mathbf{a}+2), & (\mathbf{b})(\mathbf{b}+1), & x(\mathbf{c}-1)(\mathbf{c}) \\ x(\mathbf{a}+2), & (\mathbf{b}+1), & (\mathbf{c}) \\ x^3, & x, & 1 \end{vmatrix}}{(1)\ldots(\mathbf{a}+2).(1)\ldots(\mathbf{b}+1).(1)\ldots(\mathbf{c})};$$

$$\mathrm{GF}(\infty; a, b, c, d) = \frac{\begin{vmatrix} (\mathbf{a}+1)(\mathbf{a}+2)(\mathbf{a}+3), & (\mathbf{b})(\mathbf{b}+1)(\mathbf{b}+2), & x(\mathbf{c}-1)(\mathbf{c})(\mathbf{c}+1), & x^3(\mathbf{d}-2)(\mathbf{d}-1)(\mathbf{d}) \\ x(\mathbf{a}+2)(\mathbf{a}+3), & (\mathbf{b}+1)(\mathbf{b}+2), & (\mathbf{c})(\mathbf{c}+1), & x(\mathbf{d}-1)(\mathbf{d}) \\ x^3(\mathbf{a}+3), & x(\mathbf{b}+2), & (\mathbf{c}+1), & (\mathbf{d}) \\ x^6, & x^3, & x, & 1 \end{vmatrix}}{(1)\ldots(\mathbf{a}+3).(1)\ldots(\mathbf{b}+2).(1)\ldots(\mathbf{c}+1).(1)\ldots(\mathbf{d})}.$$

In general, the determinant numerator of GF (∞ ; $a_1, a_2, \ldots, a_n$) involves x explicitly as exhibited in the determinant

$$\begin{vmatrix} 1\,, & 1\,, & x\,, & x^3\,, & \ldots, & x^{\binom{s-1}{2}} \\ x\,, & 1\,, & 1\,, & x\,, & \ldots, & x^{\binom{s-2}{2}} \\ x^3\,, & x\,, & 1\,, & 1\,, & \ldots, & x^{\binom{s-3}{2}} \\ x^6\,, & x^3\,, & x\,, & 1\,, & \ldots, & x^{\binom{s-4}{2}} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ x^{\binom{r}{2}}, & x^{\binom{r-1}{2}}, & x^{\binom{r-2}{2}}, & x^{\binom{r-3}{2}}, & \ldots, & 1 \end{vmatrix}$$

the exponents of x being figurate numbers of order 3,

$$\text{GF}\,(\infty\,;\,a^n) = \frac{1}{(\mathbf{n})\ldots(\mathbf{a+n-2}).(\mathbf{n-1})\ldots(\mathbf{a+n-3})\ldots\ldots(\mathbf{2})\ldots(\mathbf{a+1}).(\mathbf{1})\ldots(\mathbf{a})}.$$

The Restriction on the Part Magnitude.

Art. 21. I pass on to consider the case in which the part magnitude is restricted by the integer l.

I take as my point of departure the functional equation

$$(1-p_1)(1-p_2)\ldots(1-p_n)\,\text{GF}\,(l\,;\,a_1, a_2, \ldots, a_n) = x^{\Sigma a}\,\text{GF}\,(l-1\,;\,a_1, a_2, \ldots, a_n),$$

and, by means of the relation

$$\text{GF}\,(l\,;\,a_1, a_2, \ldots, a_n) = \frac{\text{L}\,(l\,;\,a_1, a_2, \ldots, a_n)}{(\mathbf{1})\,(\mathbf{2})\ldots(\mathbf{\Sigma a})},$$

convert it into a functional equation for the lattice function.

For the orders 2 and 3 we have

$$\text{L}\,(l\,;\,a, b) - x^{a+b}\,\text{L}\,(l-1\,;\,a, b)$$
$$= (\mathbf{a+b})\,\{\text{L}\,(l\,;\,a-1, b) + \text{L}\,(l\,;\,a, b-1)\} - (\mathbf{a+b-1})\,(\mathbf{a+b})\,\text{L}\,(l\,;\,a-1, b-1)\,;$$

$$\text{L}\,(l\,;\,a, b, c) - x^{a+b+c}\,\text{L}\,(l-1\,;\,a, b, c)$$
$$= (\mathbf{a+b+c})\,\{\text{L}\,(l\,;\,a-1, b, c) + \text{L}\,(l\,;\,a, b-1, c) + \text{L}\,(l\,;\,a, b, c-1)\}$$
$$-(\mathbf{a+b+c-1})(\mathbf{a+b+c})\{\text{L}(l; a-1, b-1, c) + \text{L}(l; a-1, b, c-1) + \text{L}(l; a, b-1, c-1)\}$$
$$+(\mathbf{a+b+c-2})\,(\mathbf{a+b+c-1})\,(\mathbf{a+b+c})\,\text{L}\,(l\,;\,a-1, b-1, c-1)\,;$$

and in general

$$(1-p_1\mathrm{X})(1-p_2\mathrm{X})\ldots(1-p_n\mathrm{X})\,\mathrm{L}(l\,;\,a_1, a_2, \ldots, a_n) = x^{\Sigma a}\,\mathrm{L}(l-1\,;\,a_1, a_2, \ldots, a_n)\,;$$

wherein p is the symbol of Art. 3, and symbolically

$$\mathrm{X}^m = (\Sigma\mathbf{a})(\Sigma\mathbf{a}-\mathbf{1})\ldots(\Sigma\mathbf{a}-\mathbf{m}+\mathbf{1}).$$

Art. 22. Also, from the relation

$$\mathrm{L}(l\,;\,a_1, a_2, \ldots, a_n)$$
$$= (\mathbf{1})(\mathbf{2})\ldots(\Sigma\mathbf{a})\,\frac{(\mathfrak{l}+\mathbf{n})\ldots(\mathfrak{l}+\mathbf{a}_1+\mathbf{n}-\mathbf{1}).(\mathfrak{l}+\mathbf{n}-\mathbf{1})\ldots\ldots(\mathfrak{l}+\mathbf{a}_2+\mathbf{n}-\mathbf{2})\ldots\ldots(\mathfrak{l}+\mathbf{1})\ldots(\mathfrak{l}+\mathbf{a}_n)}{(\mathbf{n})\ldots(\mathbf{a}_1+\mathbf{n}-\mathbf{1}).(\mathbf{n}-\mathbf{1})\ldots(\mathbf{a}_2+\mathbf{n}-\mathbf{2})\ldots\ldots(\mathbf{1})\ldots(\mathbf{a}_n)}\,\mathrm{IL}(l\,;\,a_1, a_2, \ldots, a_n),$$

for the orders 2 and 3 we have

$$(\mathfrak{l}+\mathbf{a}+\mathbf{1})(\mathfrak{l}+\mathbf{b})\,\mathrm{IL}(l; a,b)-(\mathbf{a}+\mathbf{1})(\mathfrak{l}+\mathbf{b})\,\mathrm{IL}(l; a-1,b)-(\mathfrak{l}+\mathbf{a}+\mathbf{1})(\mathbf{b})\,\mathrm{IL}(l; a, b-1)$$
$$+(\mathbf{a}+\mathbf{1})(\mathbf{b})\,\mathrm{IL}(l\,;\,a-1, b-1) = x^{a+b}(\mathfrak{l})(\mathfrak{l}+\mathbf{1})\,\mathrm{IL}(l-1\,;\,a, b)\,;$$

$$(\mathfrak{l}+\mathbf{a}+\mathbf{2})(\mathfrak{l}+\mathbf{b}+\mathbf{1})(\mathfrak{l}+\mathbf{c})\,\mathrm{IL}(l\,;\,a, b, c)$$
$$-(\mathbf{a}+\mathbf{2})(\mathfrak{l}+\mathbf{b}+\mathbf{1})(\mathfrak{l}+\mathbf{c})\,\mathrm{IL}(l\,;\,a-1, b, c)-(\mathfrak{l}+\mathbf{a}+\mathbf{2})(\mathbf{b}+\mathbf{1})(\mathfrak{l}+\mathbf{c})\,\mathrm{IL}(l\,;\,a, b-1, c)$$
$$-(\mathfrak{l}+\mathbf{a}+\mathbf{2})(\mathfrak{l}+\mathbf{b}+\mathbf{1})(\mathbf{c})\,\mathrm{IL}(l\,;\,a, b, c-1)$$
$$+(\mathbf{a}+\mathbf{2})(\mathbf{b}+\mathbf{1})(\mathfrak{l}+\mathbf{c})\,\mathrm{IL}(l; a-1, b-1, c)+(\mathbf{a}+\mathbf{2})(\mathfrak{l}+\mathbf{b}+\mathbf{1})(\mathbf{c})\,\mathrm{IL}(l; a-1, b, c-1)$$
$$+(\mathfrak{l}+\mathbf{a}+\mathbf{2})(\mathbf{b}+\mathbf{1})(\mathbf{c})\,\mathrm{IL}(l; a, b-1, c-1)-(\mathbf{a}+\mathbf{2})(\mathbf{b}+\mathbf{1})(\mathbf{c})\,\mathrm{IL}(l; a-1, b-1, c-1)$$
$$= x^{a+b+c}(\mathfrak{l})(\mathfrak{l}+\mathbf{1})(\mathfrak{l}+\mathbf{2})\,\mathrm{IL}(l-1\,;\,a, b, c).$$

While in general, if r_s be a symbol such that

$$r_s\,\mathrm{IL}(l\,;\,a_1, a_2, \ldots, a_n) = \frac{(\mathbf{a}_s+\mathbf{n}-\mathbf{s})}{(\mathfrak{l}+\mathbf{a}_s+\mathbf{n}-\mathbf{s})}\,\mathrm{IL}(l\,;\,a_1, a_2, \ldots, a_s-1, \ldots, a_n),$$

$$(\mathfrak{l}+\mathbf{a}_1+\mathbf{n}-\mathbf{1})(\mathfrak{l}+\mathbf{a}_2+\mathbf{n}-\mathbf{2})\ldots(\mathfrak{l}+\mathbf{a}_n)(1-r_1)(1-r_2)\ldots(1-r_n)\,\mathrm{IL}(l\,;\,a_1, a_2, \ldots, a_n)$$
$$= x^{\Sigma a}(\mathfrak{l})(\mathfrak{l}+\mathbf{1})\ldots(\mathfrak{l}+\mathbf{n}-\mathbf{1})\,\mathrm{IL}(l-1\,;\,a_1, a_2, \ldots, a_n).$$

Art. 23. I propose to obtain solutions of these functional equations. In order to ascertain the form of the required solutions it was necessary to examine several particular cases appertaining to the order 2; the result was the conjecture that

$$\mathrm{GF}(l\,;\,a, b) = \frac{(\mathfrak{l}+\mathbf{2})(\mathfrak{l}+\mathbf{3})\ldots(\mathfrak{l}+\mathbf{a}+\mathbf{1}).(\mathfrak{l}+\mathbf{1})(\mathfrak{l}+\mathbf{2})\ldots(\mathfrak{l}+\mathbf{b})}{(\mathbf{2})(\mathbf{3})\ldots(\mathbf{a}+\mathbf{1}).(\mathbf{1})(\mathbf{2})\ldots(\mathbf{b})}\left\{1+x^{b+1}\frac{(\mathbf{a}-\mathbf{b})(\mathfrak{l})}{(\mathbf{1})(\mathfrak{l}+\mathbf{a}+\mathbf{1})}\right\}.$$

This expression was found to satisfy the functional equation, so that certainly

$$\text{IL}(l\,;\,a,b) = 1+x^{b+1}\frac{(\mathfrak{a}-\mathfrak{b})(\mathfrak{l})}{(1)(\mathfrak{l}+\mathfrak{a}+1)}\,;$$

and then, observing that we may write

$$\text{IL}(l\,;\,a,b) = \frac{1}{(1)(\mathfrak{l}+\mathfrak{a}+1)}\begin{vmatrix}(\mathfrak{a}+1), & (\mathfrak{b})\\ x(\mathfrak{l}), & (\mathfrak{l}+1)\end{vmatrix},$$

and remembering the nature of solution when $l=\infty$, it became clear that we should seek solutions for the order 2 of the forms

$$\frac{(\mathfrak{a}+1)}{(\mathfrak{l}+\mathfrak{a}+1)}\,\text{F}_l,\quad \frac{(\mathfrak{b})}{(\mathfrak{l}+\mathfrak{a}+1)}\,\text{F}_l\,;$$

where F_l is a function of l to be determined in each case.

I therefore substitute $\frac{(\mathfrak{a}+1)}{(\mathfrak{l}+\mathfrak{a}+1)}\text{F}_l$ for $\text{IL}(l\,;\,a,b)$ in the functional equation and arrive at the relation

$$(\mathfrak{l})\,\text{F}_l = (\mathfrak{l}+1)\,\text{F}_{l-1}\,;$$

from which I deduce

$$\text{F}_l = (\mathfrak{l}+1),$$

yielding for me the fundamental solution

$$\frac{(\mathfrak{a}+1)(\mathfrak{l}+1)}{(\mathfrak{l}+\mathfrak{a}+1)}.$$

Similarly I find that another fundamental solution is

$$\frac{(\mathfrak{b})(\mathfrak{l})}{(\mathfrak{l}+\mathfrak{a}+1)}\,;$$

and, in terms of these two solutions, I find

$$\text{IL}(l\,;\,a,b) = \frac{1}{(1)}\left\{\frac{(\mathfrak{a}+1)(\mathfrak{l}+1)}{(\mathfrak{l}+\mathfrak{a}+1)} - x\frac{(\mathfrak{b})(\mathfrak{l})}{(\mathfrak{l}+\mathfrak{a}+1)}\right\}.$$

Art. 24. This simple exposition for the second order clearly points out the path of investigation for the third order. For, guided by the six fundamental solutions when $l=\infty$, it is natural to seek for solutions of the functional equation of the six types

$$\frac{(\mathfrak{a}+1)(\mathfrak{a}+2)(\mathfrak{b}+1)\,\text{F}_l}{(\mathfrak{l}+\mathfrak{a}+1)(\mathfrak{l}+\mathfrak{a}+2)(\mathfrak{l}+\mathfrak{b}+1)}\,;\qquad \frac{(\mathfrak{a}+1)(\mathfrak{a}+2)(\mathfrak{c})\,\text{F}_l}{(\mathfrak{l}+\mathfrak{a}+1)(\mathfrak{l}+\mathfrak{a}+2)(\mathfrak{l}+\mathfrak{b}+1)}\,;$$

$$\frac{(\mathfrak{b})(\mathfrak{b}+1)(\mathfrak{a}+2)\,\text{F}_l}{(\mathfrak{l}+\mathfrak{a}+1)(\mathfrak{l}+\mathfrak{a}+2)(\mathfrak{l}+\mathfrak{b}+1)}\,;\qquad \frac{(\mathfrak{c}-1)(\mathfrak{c})(\mathfrak{a}+2)\,\text{F}_l}{(\mathfrak{l}+\mathfrak{a}+1)(\mathfrak{l}+\mathfrak{a}+2)(\mathfrak{l}+\mathfrak{b}+1)}\,;$$

$$\frac{(\mathfrak{b})(\mathfrak{b}+1)(\mathfrak{c})\,\text{F}_l}{(\mathfrak{l}+\mathfrak{a}+1)(\mathfrak{l}+\mathfrak{a}+2)(\mathfrak{l}+\mathfrak{b}+1)}\,;\qquad \frac{(\mathfrak{c}-1)(\mathfrak{c})(\mathfrak{b}+1)\,\text{F}_l}{(\mathfrak{l}+\mathfrak{a}+1)(\mathfrak{l}+\mathfrak{a}+2)(\mathfrak{l}+\mathfrak{b}+1)}\,;$$

where F_l is a function of l to be determined in each case.

Substituting the first of these for $\mathrm{IL}(l; a, b, c)$ in the functional equation of order 3 I find

$$F_l = \frac{(l+1)(l+2)}{(l)^2} F_{l-1};$$

but the solution of the equation

$$F_l = \phi_l F_{l-1}$$

is clearly

$$F_l = \phi_l \phi_{l-1} \phi_{l-2} \ldots ;$$

so that, in the present case,

$$F_l = \frac{(l+1)(l+2)}{(l)^2} \cdot \frac{(l)(l+1)}{(l-1)^2} \cdot \frac{(l-1)(l)}{(l-2)^2} \ldots = (l+1)^2 (l+2);$$

so that I obtain a fundamental solution

$$\frac{(a+1)(a+2)(b+1)(l+1)^2(l+2)}{(l+a+1)(l+a+2)(l+b+1)}.$$

Art. 25. Similarly I arrive at five other fundamental solutions,

$$\frac{(a+1)(a+2)(c)(l)(l+1)(l+2)}{(l+a+1)(l+a+2)(l+b+1)},$$

$$\frac{(b)(b+1)(a+2)(l)(l+1)(l+2)}{(l+a+1)(l+a+2)(l+b+1)},$$

$$\frac{(c-1)(c)(a+2)(l)^2(l+1)}{(l+a+1)(l+a+2)(l+b+1)},$$

$$\frac{(b)(b+1)(c)(l-1)(l)(l+2)}{(l+a+1)(l+a+2)(l+b+1)},$$

$$\frac{(c-1)(c)(b+1)(l-1)(l)(l+1)}{(l+a+1)(l+a+2)(l+b+1)};$$

and I next seek, guided by previous work, to construct the function $\mathrm{IL}(l; a, b, c)$ by a linear function of these six solutions.

It is natural to write

$$\begin{aligned}&(a+1)(a+2)(b+1)(l+1)^2(l+2) - x(b)(b+1)(a+2)(l)(l+1)(l+2)\\ &-x(a+1)(a+2)(c)(l)(l+1)(l+2) + x^3(c-1)(c)(a+2)(l)^2(l+1)\\ &+x^3(b)(b+1)(c)(l-1)(l)(l+2) - x^4(c-1)(c)(b+1)(l-1)(l)(l+1),\end{aligned}$$

with a denominator

$$(1)^2(2)(l+a+1)(l+a+2)(l+b+1),$$

which, in determinant form, is

$$\begin{vmatrix} (\mathrm{a}+1)(\mathrm{a}+2), & (\mathrm{b})(\mathrm{b}+1)\;, & x(\mathrm{c}-1)(\mathrm{c}) \\ x(\mathrm{a}+2)(\mathrm{l})\;, & (\mathrm{b}+1)(\mathrm{l}+1), & (\mathrm{c})(\mathrm{l}+2) \\ x^3(\mathrm{l}-1)(\mathrm{l})\;, & x(\mathrm{l})(\mathrm{l}+1)\;, & (\mathrm{l}+1)(\mathrm{l}+2) \end{vmatrix}$$

divided by

$$(1)^2(2)(\mathrm{l}+\mathrm{a}+1)(\mathrm{l}+\mathrm{a}+2)(\mathrm{l}+\mathrm{b}+1)\,;$$

and it will be shown that it is, in fact, the expression of IL $(l\,;\,a,b,c)$.

Art. 26. In a general manner we may take the following view—

Recalling a previous notation for the order 3

$$\mathrm{A}_2=(\mathrm{a}+1)(\mathrm{a}+2),\quad \mathrm{A}_1=(\mathrm{a}+2),\quad \mathrm{B}_2=(\mathrm{b})(\mathrm{b}+1),\quad \mathrm{B}_1=(\mathrm{b}+1),\quad \mathrm{C}_2=(\mathrm{c}-1)(\mathrm{c})$$

$$\mathrm{C}_1=(\mathrm{c}),\quad \mathrm{A}_0=\mathrm{B}_0=\mathrm{C}_0=1\,;$$

and taking a product

$$\mathrm{A}_p\mathrm{B}_q\mathrm{C}_r,$$

where p, q, r denotes some permutation of the numbers 2, 1, 0, suppose

$$\frac{\mathrm{A}_p\mathrm{B}_q\mathrm{C}_r\mathrm{F}_l}{(\mathrm{l}+\mathrm{a}+1)(\mathrm{l}+\mathrm{a}+2)(\mathrm{l}+\mathrm{b}+1)}$$

substituted for IL $(l\,;\,a,b,c)$ is the functional equation. The result on reduction is

$$\mathrm{F}_l=\frac{(\mathrm{l})(\mathrm{l}+1)(\mathrm{l}+2)}{(\mathrm{l}-2+\mathrm{p})(\mathrm{l}-1+\mathrm{q})(\mathrm{l}+\mathrm{r})}\mathrm{F}_{l-1}\,;$$

and now writing

$$\frac{(\mathrm{l})}{(\mathrm{l}-2+\mathrm{p})}\cdot\frac{(\mathrm{l}-1)}{(\mathrm{l}-3+\mathrm{p})}\cdot\frac{(\mathrm{l}-2)}{(\mathrm{l}-4+\mathrm{p})}\cdot\ldots\textit{ ad inf.}=\phi_{ap}\,,$$

$$\frac{(\mathrm{l}+1)}{(\mathrm{l}-1+\mathrm{q})}\cdot\frac{(\mathrm{l})}{(\mathrm{l}-2+\mathrm{q})}\cdot\frac{(\mathrm{l}-1)}{(\mathrm{l}-3+\mathrm{q})}\cdot\ldots\textit{ ad inf.}=\phi_{bq}\,,$$

$$\frac{(\mathrm{l}+2)}{(\mathrm{l}+\mathrm{r})}\cdot\frac{(\mathrm{l}+1)}{(\mathrm{l}-1+\mathrm{r})}\cdot\frac{(\mathrm{l})}{(\mathrm{l}-2+\mathrm{r})}\cdot\ldots\textit{ ad inf.}=\phi_{cr}\,;$$

$$\mathrm{F}_l=\phi_{ap}\phi_{bq}\phi_{cr}\,;$$

and the inner lattice function IL $(l\,;\,a,b,c)$ is, to a divisor

$$(1)^2(2)(\mathrm{l}+\mathrm{a}+1)(\mathrm{l}+\mathrm{a}+2)(\mathrm{l}+\mathrm{b}+1)$$

près, equal to the determinant

$$\begin{vmatrix} A_2\phi_{a2}\,, & B_2\phi_{b2}\,, & xC_2\phi_{c2} \\ xA_1\phi_{a1}\,, & B_1\phi_{b1}\,, & C_1\phi_{c1} \\ x^3A_0\phi_{a0}, & xB_0\phi_{b0}, & C_0\phi_{c0} \end{vmatrix},$$

and this determinant is clearly

$$\begin{vmatrix} A_2 & , & B_2 & , & xC_2 \\ xA_1(l) & , & B_1(l+1) & , & C_1(l+2) \\ x^3A_0(l-1)(l), & & xB_0(l)(l+1), & & C_0(l+1)(l+2) \end{vmatrix}.$$

Art. 27. This is evidently a perfectly general process and suffices to establish that a solution of the functional equation of order n is a determinant of order n of which the constituent in the s^{th} row and t^{th} column is

$$x^{\binom{s-t+1}{2}}(a_t+s-t+1)\ldots(a_t+n-t).(l-s+t+1)\ldots(l+t-1);$$

and when this determinant is divided by

$$(1)^{n-1}(2)^{n-2}\ldots(n+1)(l+a_1+1)\ldots(l+a_1+n-1).(l+a_2+1)\ldots(l+a_2+n-2)\ldots(l+a_{n-1}+1),$$

we have, as will be proved, the expression of

$$\text{IL}\,(l\,;\,a_1, a_2, \ldots, a_n).$$

Art. 28. To establish this we may apply a series of tests.

Thus, take the expression of IL $(l\,;\,a, b, c, d)$

$$\begin{vmatrix} (a+1)(a+2)(a+3), & (b)(b+1)(b+2)\,, & x(c-1)(c)(c+1), & x^3(d-2)(d-1)(d) \\ x(l)(a+2)(a+3)\,, & (l+1)(b+1)(b+2), & (l+2)(c)(c+1)\,, & x(l+3)(d-1)(d) \\ x^3(l-1)(l)(a+3)\,, & x(l)(l+1)(b+2)\,, & (l+1)(l+2)(c+1), & (l+2)(l+3)(d) \\ x^6(l-2)(l-1)(l)\,, & x^3(l-1)(l)(l+1)\,, & x(l)(l+1)(l+2)\,, & (l+1)(l+2)(l+3) \end{vmatrix}$$

divided by

$$(1)^3(2)^2(3)(l+a+1)(l+a+2)(l+a+3)(l+b+1)(l+b+2)(l+c+1).$$

It clearly reduces to IL $(\infty\,;\,a, b, c, d)$ when l is put equal to ∞.

Moreover, it can be shown that when $d=c=b=a$, the determinant involves l and a always in the combination $l+a$; for consider the determinant in question when l is put equal to $-a$.

It is

$$\left|\begin{array}{cccc} (\mathbf{a}+1)(\mathbf{a}+2)(\mathbf{a}+3), & (\mathbf{a})(\mathbf{a}+1)(\mathbf{a}+2), & x(\mathbf{a}-1)(\mathbf{a})(\mathbf{a}+1), & x^3(\mathbf{a}-2)(\mathbf{a}-1)(\mathbf{a}) \\ x(-\mathbf{a})(\mathbf{a}+2)(\mathbf{a}+3), & (-\mathbf{a}+1)(\mathbf{a}+1)(\mathbf{a}+2), & (-\mathbf{a}+2)(\mathbf{a})(\mathbf{a}+1), & x(-\mathbf{a}+3)(\mathbf{a}-1)(\mathbf{a}) \\ x^3(-\mathbf{a}-1)(-\mathbf{a})(\mathbf{a}+3), & x(-\mathbf{a})(-\mathbf{a}+1)(\mathbf{a}+2), & (-\mathbf{a}+1)(-\mathbf{a}+2)(\mathbf{a}+1), & (-\mathbf{a}+2)(-\mathbf{a}+3)(\mathbf{a}) \\ x^6(-\mathbf{a}-2)(-\mathbf{a}-1)(-\mathbf{a}), & x^3(-\mathbf{a}-1)(-\mathbf{a})(-\mathbf{a}+1), & x(-\mathbf{a})(-\mathbf{a}+1)(-\mathbf{a}+2), & (-\mathbf{a}+1)(-\mathbf{a}+2)(-\mathbf{a}+3) \end{array}\right|,$$

and since $(-\mathbf{s}) = -x^{-s}(\mathbf{s})$, it may be written

$$x^{10-6a}\left|\begin{array}{cccc} (\mathbf{a}+1)(\mathbf{a}+2)(\mathbf{a}+3), & (\mathbf{a})(\mathbf{a}+1)(\mathbf{a}+2), & (\mathbf{a}-1)(\mathbf{a})(\mathbf{a}+1), & (\mathbf{a}-2)(\mathbf{a}-1)(\mathbf{a}) \\ (\mathbf{a})(\mathbf{a}+2)(\mathbf{a}+3), & (\mathbf{a}-1)(\mathbf{a}+1)(\mathbf{a}+2), & (\mathbf{a}-2)(\mathbf{a})(\mathbf{a}+1), & (\mathbf{a}-3)(\mathbf{a}-1)(\mathbf{a}) \\ (\mathbf{a})(\mathbf{a}+1)(\mathbf{a}+3), & (\mathbf{a}-1)(\mathbf{a})(\mathbf{a}+2), & (\mathbf{a}-2)(\mathbf{a}-1)(\mathbf{a}+1), & (\mathbf{a}-3)(\mathbf{a}-2)(\mathbf{a}) \\ (\mathbf{a})(\mathbf{a}+1)(\mathbf{a}+2), & (\mathbf{a}-1)(\mathbf{a})(\mathbf{a}+1), & (\mathbf{a}-2)(\mathbf{a}-1)(\mathbf{a}), & (\mathbf{a}-3)(\mathbf{a}-2)(\mathbf{a}-1) \end{array}\right|;$$

and now, putting $-a$ for a, we find after a few transformations by multiplication and division of columns and rows that it is unchanged; this suffices to show that the expression is not a function of a at all since it is clearly not of the form $k+\phi(a)+\phi(-a)$. Hence the determinant under examination is a function of $l+a$. Put therein $l = p-a$ and then put $a = 0$; we find

$$\left|\begin{array}{cccc} (1)(2)(3), & 0, & 0, & 0 \\ x(\mathbf{p})(2)(3), & (\mathbf{p}+1)(1)(2), & 0, & 0 \\ x^3(\mathbf{p}-1)(\mathbf{p})(3), & x(\mathbf{p})(\mathbf{p}+1)(2), & (\mathbf{p}+1)(\mathbf{p}+2)(1), & 0 \\ x^6(\mathbf{p}-2)(\mathbf{p}-1)(\mathbf{p}), & x^3(\mathbf{p}-1)(\mathbf{p})(\mathbf{p}+1), & x(\mathbf{p})(\mathbf{p}+1)(\mathbf{p}+2), & (\mathbf{p}+1)(\mathbf{p}+2)(\mathbf{p}+3) \end{array}\right|,$$

which has the value

$$(1)^3(2)^2(3)(\mathfrak{l}+\mathbf{a}+1)^3(\mathfrak{l}+\mathbf{a}+2)^2(\mathfrak{l}+\mathbf{a}+3),$$

and thus

$$\mathrm{IL}(l;\, a^4) = 1.$$

This is a verification because in all particular cases of this nature that have been examined the inner lattice function is unity.

Art. 29. We may now resume the results.

$$\mathrm{GF}\,(l\,;\,ab)$$

$$=\frac{(1)\ldots(\mathfrak{l}+\mathfrak{a})}{(1)\ldots(\mathfrak{l}+1)\,.\,(1)\ldots(\mathfrak{a}+1)}\cdot\frac{(1)\ldots(\mathfrak{l}+\mathfrak{b})}{(1)\ldots(\mathfrak{l})\,.\,(1)\ldots(\mathfrak{b})}\cdot\begin{vmatrix}(\mathfrak{a}+1), & (\mathfrak{b})\\ x(\mathfrak{l})\ , & (\mathfrak{l}+1)\end{vmatrix};$$

$$\mathrm{GF}\,(l\,;\,a^2)$$

$$=\frac{(\mathfrak{l}+2)\ldots(\mathfrak{l}+\mathfrak{a}+1)}{(2)\ldots(\mathfrak{a}+1)}\cdot\frac{(\mathfrak{l}+1)\ldots(\mathfrak{l}+\mathfrak{a})}{(1)\ldots(\mathfrak{a})};$$

$$\mathrm{GF}\,(l\,;\,abc)$$

$$=\frac{(1)\ldots(\mathfrak{l}+\mathfrak{a})}{(1)\ldots(\mathfrak{l}+2).(1)\ldots(\mathfrak{a}+2)}\cdot\frac{(1)\ldots(\mathfrak{l}+\mathfrak{b})}{(1)\ldots(\mathfrak{l}+1).(1)\ldots(\mathfrak{b}+1)}\cdot\frac{(1)\ldots(\mathfrak{l}+\mathfrak{c})}{(1)\ldots(\mathfrak{l}).(1)\ldots(\mathfrak{c})}\cdot\begin{vmatrix}(\mathfrak{a}+1)(\mathfrak{a}+2), & (\mathfrak{b})(\mathfrak{b}+1)\ , & x(\mathfrak{c}-1)(\mathfrak{c})\\ x(\mathfrak{l})(\mathfrak{a}+2)\ , & (\mathfrak{l}+1)(\mathfrak{b}+1), & (\mathfrak{l}+2)(\mathfrak{c})\\ x^3(\mathfrak{l}-1)(\mathfrak{l})\ , & x(\mathfrak{l})(\mathfrak{l}+1)\ , & (\mathfrak{l}+1)(\mathfrak{l}+2)\end{vmatrix};$$

$$\mathrm{GF}\,(l\,;\,a^3)$$

$$=\frac{(\mathfrak{l}+3)\ldots(\mathfrak{l}+\mathfrak{a}+2)}{(3)\ldots(\mathfrak{a}+2)}\cdot\frac{(\mathfrak{l}+2)\ldots(\mathfrak{l}+\mathfrak{a}+1)}{(2)\ldots(\mathfrak{a}+1)}\cdot\frac{(\mathfrak{l}+1)\ldots(\mathfrak{l}+\mathfrak{a})}{(1)\ldots(\mathfrak{a})};$$

$$\mathrm{GF}\,(l\,;\,a,b,c,d)$$

$$=\frac{(1)\ldots(\mathfrak{l}+\mathfrak{a})}{(1)\ldots(\mathfrak{l}+3)\,.\,(1)\ldots(\mathfrak{a}+3)}\cdot\frac{(1)\ldots(\mathfrak{l}+\mathfrak{b})}{(1)\ldots(\mathfrak{l}+2)\,.\,(1)\ldots(\mathfrak{b}+2)}\cdot\frac{(1)\ldots(\mathfrak{l}+\mathfrak{c})}{(1)\ldots(\mathfrak{l}+1)\,.\,(1)\ldots(\mathfrak{c}+1)}\cdot\frac{(1)\ldots(\mathfrak{l}+\mathfrak{d})}{(1)\ldots(\mathfrak{l})\,.\,(1)\ldots(\mathfrak{d})}$$

$$\times\begin{vmatrix}(\mathfrak{a}+1)(\mathfrak{a}+2)(\mathfrak{a}+3), & (\mathfrak{b})(\mathfrak{b}+1)(\mathfrak{b}+2)\ , & x(\mathfrak{c}-1)(\mathfrak{c})(\mathfrak{c}+1)\ , & x^3(\mathfrak{d}-2)(\mathfrak{d}-1)(\mathfrak{d})\\ x(\mathfrak{l})(\mathfrak{a}+2)(\mathfrak{a}+3)\ , & (\mathfrak{l}+1)(\mathfrak{b}+1)(\mathfrak{b}+2), & (\mathfrak{l}+2)(\mathfrak{c})(\mathfrak{c}+1)\ , & x(\mathfrak{l}+3)(\mathfrak{d}-1)(\mathfrak{d})\\ x^3(\mathfrak{l}-1)(\mathfrak{l})(\mathfrak{a}+3)\ , & x(\mathfrak{l})(\mathfrak{l}+1)(\mathfrak{b}+2)\ , & (\mathfrak{l}+1)(\mathfrak{l}+2)(\mathfrak{c}+1), & (\mathfrak{l}+2)(\mathfrak{l}+3)(\mathfrak{d})\\ x^6(\mathfrak{l}-2)(\mathfrak{l}-1)(\mathfrak{l})\ , & x^3(\mathfrak{l}-1)(\mathfrak{l})(\mathfrak{l}+1)\ , & x(\mathfrak{l})(\mathfrak{l}+1)(\mathfrak{l}+2)\ , & (\mathfrak{l}+1)(\mathfrak{l}+2)(\mathfrak{l}+3)\end{vmatrix};$$

and so forth, the law of formation being quite clear.

It is a remarkable fact that this elegant result appears to be valid whatsoever the equalities may be that present themselves between the integers $a, b, c, d, \ldots$. The discussion of this and the interpretation of GF $(l\,;\,a, b, c, d, \ldots)$ when $a, b, c, d, \ldots$, are not in descending order must be deferred to another occasion.

I add the general result for equal rows which has now been established

$$\text{GF}\,(l\,;\,a^n)$$
$$= \frac{(l+n)\ldots(l+a+n-1)}{(n)\ldots(a+n-1)} \cdot \frac{(l+n-1)\ldots(l+a+n-2)}{(n-1)\ldots(a+n-2)} \cdots\cdots \frac{(l+1)\ldots(l+a)}{(1)\ldots(a)}.$$

Art. 30. I recall that, in Part V., the formula was given

$$\text{GF}\,(l\,;\,a, b, c, d, \ldots) = \text{L}_0\text{GF}\,(l\,;\,\Sigma a) + \text{L}_1\text{GF}\,(l-1\,;\,\Sigma a) + \ldots + \text{L}_\mu\text{GF}\,(l-\mu\,;\,\Sigma a),$$

wherein L_s is the sub-lattice function of order s derived from the lattice whose specification is $(abcd\ldots)$. So far it has not been feasible to directly determine an expression for L_s, but as we now know the expression for GF $(l\,;\,a, b, c, d, \ldots)$ it is possible to find expressions for $\text{L}_1, \text{L}_2, \text{L}_3, \ldots$, by giving l the values $1, 2, 3, \ldots$, in succession in the resulting identity. One interesting result was, however, directly determined, viz. :—

$$\text{L}_s\,(\infty\,;\,a, b)$$
$$= x^{s(s+1)} \frac{(a-s+1)\ldots(a-1)}{(1)\ldots(s)} \cdot \frac{(b-s+1)\ldots(b)}{(1)\ldots(s+1)} \{(1)(a-s) + x^{b-s+1}(a-b)(s)\}\,;$$

but I do not give the proof of it at present, as the subject of the sub-lattice functions has not yet been worked out, and they are, in fact, no longer necessary for this part of the general investigation.

Art. 31. Valuable information, concerning line or one-dimensional partitions, is furnished by putting $l = 1$ in the general formula.

The partitions that are then enumerated are those in which every part is unity, there being not more than a_s units in the s^{th} row; if thence we proceed to line partitions by adding the units that appear in each row we clearly get a system of line partitions such that the s^{th} part is limited in magnitude by the integer a_s; or the system comprises all partitions contained in or subordinate to $(a_1, a_2, \ldots, a_n)$, viz., such that the first part $\ngtr a_1$, the second $\ngtr a_2, \ldots$, the $n^{\text{th}} \ngtr a_n$.

Denoting the generating function of these by

$$\text{GF}\,(a_1, a_2, \ldots, a_n\,;\,n),$$

and, denoting $\dfrac{(1)(2)\ldots(p)}{(1)(2)\ldots(q)\,.\,(1)(2)\ldots(p-q)}$ by X_{pq} or $\text{X}_{p,p-q}$, I find

$$\text{GF}\,(a, b\,;\,2) = \frac{(b+1)}{(1)(2)} \cdot \begin{vmatrix} (a+1), & (b) \\ x\;\;, & \text{X}_{21} \end{vmatrix}\,;$$

$$\mathrm{GF}(a, b, c\,;\, 3) = \frac{(\mathbf{c}+1)}{(1)(2)(3)} \begin{vmatrix} (\mathbf{a}+1), & (\mathbf{b})(\mathbf{b}+1), & x(\mathbf{c}-1)(\mathbf{c}) \\ x\;, & \mathrm{X}_{21}(\mathbf{b}+1), & \mathrm{X}_{31}(\mathbf{c}) \\ 0\;, & x\;, & \mathrm{X}_{32} \end{vmatrix};$$

$$\mathrm{GF}(a, b, c, d\,;\, 4)$$

$$= \frac{(\mathbf{d}+1)}{(1)\ldots(4)} \begin{vmatrix} (\mathbf{a}+1), & (\mathbf{b})(\mathbf{b}+1), & x(\mathbf{c}-1)(\mathbf{c})(\mathbf{c}+1), & x^3(\mathbf{d}-2)(\mathbf{d}-1)(\mathbf{d}) \\ x\;, & \mathrm{X}_{21}(\mathbf{b}+1), & \mathrm{X}_{31}(\mathbf{c})(\mathbf{c}+1)\;, & x\mathrm{X}_{41}(\mathbf{d}-1)(\mathbf{d}) \\ 0\;, & x\;, & \mathrm{X}_{32}(\mathbf{c}+1)\;, & \mathrm{X}_{42}(\mathbf{d}) \\ 0\;, & 0\;, & x\;, & \mathrm{X}_{43} \end{vmatrix};$$

$$\mathrm{GF}(a, b, c, d, e\,;\, 5)$$

$$= \frac{(\mathbf{e}+1)}{(1)\ldots(5)} \begin{vmatrix} (\mathbf{a}+1), & (\mathbf{b})(\mathbf{b}+1), & x(\mathbf{c}-1)(\mathbf{c})(\mathbf{c}+1), & x^3(\mathbf{d}-2)(\mathbf{d}-1)(\mathbf{d})(\mathbf{d}+1), & x^6(\mathbf{e}-3)(\mathbf{e}-2)(\mathbf{e}-1)(\mathbf{e}) \\ x\;, & \mathrm{X}_{21}(\mathbf{b}+1), & \mathrm{X}_{31}(\mathbf{c})(\mathbf{c}+1)\;, & x\mathrm{X}_{41}(\mathbf{d}-1)(\mathbf{d})(\mathbf{d}+1)\;, & x^3\mathrm{X}_{51}(\mathbf{e}-2)(\mathbf{e}-1)(\mathbf{e}) \\ 0\;, & x\;, & \mathrm{X}_{32}(\mathbf{c}+1),\;, & \mathrm{X}_{42}(\mathbf{d})(\mathbf{d}+1)\;, & x\mathrm{X}_{52}(\mathbf{e}-1)(\mathbf{e}) \\ 0\;, & 0\;, & x\;, & \mathrm{X}_{43}(\mathbf{d}+1)\;, & \mathrm{X}_{53}(\mathbf{e}) \\ 0\;, & 0\;, & 0\;, & x\;, & \mathrm{X}_{54} \end{vmatrix};$$

and the law is evident.

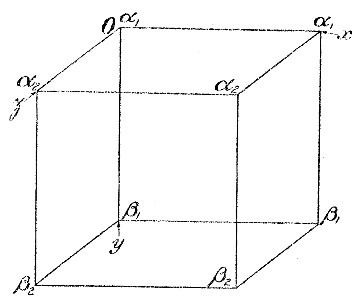
Art. 32. In general, supposing the lattice to be in the plane of xy, that of the paper and the axis of z perpendicular to the plane of the paper, if we project the partition on to the plane of yz, we obtain a partition at the nodes of a lattice of l rows in which the part magnitude in the s^{th} columns is limited by the number α_s.

The general formula for $\mathrm{GF}(l\,;\, a_1, a_2, \ldots, a_n)$ is remarkable from the fact that

$$\mathrm{GF}(l\,;\, a_1, a_2, \ldots, a_n) = \mathrm{GF}(l\,;\, b_1, b_2, \ldots, b_m),$$

where $(a_1, a_2, \ldots, a_n)$, $(b_1, b_2, \ldots, b_m)$ are any two conjugate line partitions.

Adumbration of the Three-dimensional Theory.

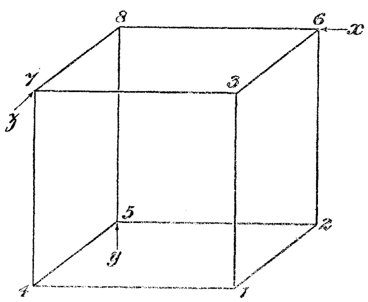


Art. 33. I conclude this Part by pointing out a path of future investigation into the Theory of Partitions in space of three dimensions.

I consider a complete or incomplete lattice in three dimensions, the lines of the lattice being in the direction of three rectangular axes of x, y, z respectively. Just as an incomplete lattice in two dimensions is defined by a one-dimensional partition whose successive parts specify the successive rows of the lattice, so an incomplete lattice in three dimensions is defined by a two-dimensional partition whose successive rows specify the successive layers of the lattice.

I shall suppose these layers to be in or parallel to the plane of xy which is the plane of the paper, and the axis of z to be perpendicular to the plane of the paper. Descending order of magnitude of parts placed at the points of the lattice is to be in evidence in the three directions Ox, Oy, Oz.

Art. 34. Consider the simplest case of a complete lattice, the points forming the summits of a cube. The two-dimensional lattices $\alpha_1\alpha_1\beta_1\beta_1$, $\alpha_2\alpha_2\beta_2\beta_2$, in and parallel to the plane of the paper are superposed to form the three-dimensional lattice.



Suppose that the first 8 integers are placed at the points of the lattice so that descending order of magnitude is in evidence in the directions Ox, Oy, Oz, *e.g.*, one of 48 such arrangements is as shown.

I associate with the first and second rows of the first layer the letters α_1, β_1 respectively, and with the first and second rows of the second layer the letters α_2, β_2 respectively, and then from the illustrated arrangement of the first 8 numbers I derive a Greek-letter succession in the following manner :—I take the numbers in descending order of magnitude and write down the Greek letter with which the *position* of each number is associated : thus the arrangement above gives

$$\begin{matrix} 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ \alpha_1 & \alpha_2 & \alpha_1 & \beta_1 & \beta_2 & \alpha_2 & \beta_1 & \beta_2. \end{matrix}$$

Art. 35. In this Greek-letter succession we have to *note*

(i.) A β which is succeeded by an α,

(ii.) An α which is succeeded by an α with a smaller suffix,

(iii.) A β which is succeeded by a β with a smaller suffix.

If a letter which is thus noted is the s^{th} letter in the permutation I associate with the permutation the power $x^{\Sigma s}$, and taking the sum of these powers in respect of the whole of the permutations associated with and derived from the lattice I obtain the lattice function

$$\Sigma x^{\Sigma s};$$

and, following the reasoning of Part V., Art. 6, I derive the generating function for partitions at the points of the lattice, the part magnitude being unrestricted, viz.,

$$\text{GF}(\infty\,;\,22\,;\,22) = \frac{\Sigma x^{\Sigma s}}{(1)(2)\ldots(8)} = \frac{\text{L}(\infty\,;\,22\,;\,22)}{(1)(2)\ldots(8)}.$$

Similarly, from the Greek-letter successions which involve *t noted* letters, I derive the sub-lattice function of order t, and thence, by previous reasoning, arrive at the generating function when the part magnitude is restricted by the integer l,

$$\text{GF}(l;\,22;\,22) = \frac{\Sigma L_t(\infty;\,22;\,22)\,(l-t+1)\,(l-t+2)\ldots(l-t+8)}{(1)\,(2)\ldots(8)}.$$

Art. 36. The method is generally applicable to any incomplete lattice in three dimensions. I work out in detail the case in which the points of the lattice form the 8 summits of a cube, in order to show that the result obtained, in Part II., Section 7, in quite a different manner, is verified. That result, with modified notation, was

Generating Function

$$= \frac{(l+1)\ldots(l+8)}{(1)\ldots(8)} + \text{P}(x)\frac{(l)\ldots(l+7)}{(1)\ldots(8)} + \text{Q}(x)\frac{(l-1)\ldots(l+6)}{(1)\ldots(8)}$$
$$+\text{R}(x)\frac{(l-2)\ldots(l+5)}{(1)\ldots(8)} + x^{16}\frac{(l-3)\ldots(l+4)}{(1)\ldots(8)};$$

where

$$\text{P}(x) = 2x^2+2x^3+3x^4+2x^5+2x^6,$$

$$\text{Q}(x) = x^5+3x^6+4x^7+8x^8+4x^9+3x^{10}+x^{11},$$

$$\text{R}(x) = 2x^{10}+2x^{11}+3x^{12}+2x^{13}+2x^{14}.$$

I shall now show that

$$1, \quad \text{P}(x), \quad \text{Q}(x), \quad \text{R}(x), \quad x^{16}$$

are, in fact, the sub-lattice functions of orders 0 to 4 which appertain to the lattice formed by the summits of a cube.

I write down the 48 permutations of the Greek letters and over each the arrangement of the first 8 integers from which it is derived, the lower layer of numbers being placed to the left :—

$\begin{matrix}87&65\\43&21\end{matrix}$, $\alpha_1\alpha_1\alpha_2\alpha_2\beta_1\beta_1\beta_2\beta_2$,	$\begin{matrix}85&64\\73&21\end{matrix}$, $\alpha_1\beta_1 \mid \alpha_2 \mid \alpha_1\alpha_2\beta_1\beta_2\beta_2$,	$\begin{matrix}85&73\\64&21\end{matrix}$, $\alpha_1\alpha_2\beta_1 \mid \alpha_1\beta_1 \mid \alpha_2\beta_2\beta_2$,	$\begin{matrix}85&63\\74&21\end{matrix}$, $\alpha_1\beta_1 \mid \alpha_2 \mid \alpha_1\beta_1 \mid \alpha_2\beta_2\beta_2$,
$\begin{matrix}86&75\\43&21\end{matrix}$, $\alpha_1\alpha_2 \mid \alpha_1\alpha_2\beta_1\beta_1\beta_2\beta_2$,	$\begin{matrix}86&74\\53&21\end{matrix}$, $\alpha_1\alpha_2 \mid \alpha_1\beta_1 \mid \alpha_2\beta_1\beta_2\beta_2$,	$\begin{matrix}85&73\\62&41\end{matrix}$, $\alpha_1\alpha_2\beta_1 \mid \alpha_1\beta_2 \mid \alpha_2\beta_1\beta_2$,	$\begin{matrix}85&63\\72&41\end{matrix}$, $\alpha_1\beta_1 \mid \alpha_2 \mid \alpha_1\beta_2 \mid \alpha_2\beta_1\beta_2$,
$\begin{matrix}86&54\\73&21\end{matrix}$, $\alpha_1\beta_1 \mid \alpha_1\alpha_2\alpha_2\beta_1\beta_2\beta_2$,	$\begin{matrix}86&43\\75&21\end{matrix}$, $\alpha_1\beta_1 \mid \alpha_1\beta_1 \mid \alpha_2\alpha_2\beta_2\beta_2$,	$\begin{matrix}86&54\\72&31\end{matrix}$, $\alpha_1\beta_1 \mid \alpha_1\alpha_2\alpha_2\beta_2 \mid \beta_1\beta_2$,	$\begin{matrix}85&64\\72&31\end{matrix}$, $\alpha_1\beta_1 \mid \alpha_2 \mid \alpha_1\alpha_2\beta_2 \mid \beta_1\beta_2$,
$\begin{matrix}87&54\\63&21\end{matrix}$, $\alpha_1\alpha_1\beta_1 \mid \alpha_2\alpha_2\beta_1\beta_2\beta_2$,	$\begin{matrix}84&63\\72&51\end{matrix}$, $\alpha_1\beta_1 \mid \alpha_2\beta_2 \mid \alpha_1\alpha_2\beta_1\beta_2$,	$\begin{matrix}86&52\\74&31\end{matrix}$, $\alpha_1\beta_1 \mid \alpha_1\alpha_2\beta_1\beta_2 \mid \alpha_2\beta_2$,	$\begin{matrix}85&62\\74&31\end{matrix}$, $\alpha_1\beta_1 \mid \alpha_2 \mid \alpha_1\beta_1\beta_2 \mid \alpha_2\beta_2$,
$\begin{matrix}85&74\\63&21\end{matrix}$, $\alpha_1\alpha_2\beta_1 \mid \alpha_1\alpha_2\beta_1\beta_2\beta_2$,	$\begin{matrix}86&73\\54&21\end{matrix}$, $\alpha_1\alpha_2 \mid \alpha_1\beta_1\beta_1 \mid \alpha_2\beta_2\beta_2$,	$\begin{matrix}87&54\\62&31\end{matrix}$, $\alpha_1\alpha_1\beta_1 \mid \alpha_2\alpha_2\beta_2 \mid \beta_1\beta_2$,	$\begin{matrix}86&74\\52&31\end{matrix}$, $\alpha_1\alpha_2 \mid \alpha_1\beta_1 \mid \alpha_2\beta_2 \mid \beta_1\beta_2$,
$\begin{matrix}87&64\\53&21\end{matrix}$, $\alpha_1\alpha_1\alpha_2\beta_1 \mid \alpha_2\beta_1\beta_2\beta_2$,	$\begin{matrix}86&73\\52&41\end{matrix}$, $\alpha_1\alpha_2 \mid \alpha_1\beta_1\beta_2 \mid \alpha_2\beta_1\beta_2$,	$\begin{matrix}87&52\\64&31\end{matrix}$, $\alpha_1\alpha_1\beta_1 \mid \alpha_2\beta_1\beta_2 \mid \alpha_2\beta_2$,	$\begin{matrix}86&42\\75&31\end{matrix}$, $\alpha_1\beta_1 \mid \alpha_1\beta_1 \mid \alpha_2\beta_2 \mid \alpha_2\beta_2$,
$\begin{matrix}87&43\\65&21\end{matrix}$, $\alpha_1\alpha_1\beta_1\beta_1 \mid \alpha_2\alpha_2\beta_2\beta_2$,	$\begin{matrix}86&53\\74&21\end{matrix}$, $\alpha_1\beta_1 \mid \alpha_1\alpha_2\beta_1 \mid \alpha_2\beta_2\beta_2$,	$\begin{matrix}85&74\\62&31\end{matrix}$, $\alpha_1\alpha_2\beta_1 \mid \alpha_1\alpha_2\beta_2 \mid \beta_1\beta_2$,	$\begin{matrix}84&62\\73&51\end{matrix}$; $\alpha_1\beta_1 \mid \alpha_2\beta_2 \mid \alpha_1\beta_1 \mid \alpha_2\beta_2$,
$\begin{matrix}84&73\\62&51\end{matrix}$, $\alpha_1\alpha_2\beta_1\beta_2 \mid \alpha_1\alpha_2\beta_1\beta_2$,	$\begin{matrix}86&53\\72&41\end{matrix}$, $\alpha_1\beta_1 \mid \alpha_1\alpha_2\beta_2 \mid \alpha_2\beta_1\beta_2$,	$\begin{matrix}85&72\\64&31\end{matrix}$, $\alpha_1\alpha_2\beta_1 \mid \alpha_1\beta_1\beta_2 \mid \alpha_2\beta_2$,	$\begin{matrix}86&72\\53&41\end{matrix}$, $\alpha_1\alpha_2 \mid \alpha_1\beta_1\beta_2 \mid \beta_1 \mid \alpha_2\beta_2$,
$\begin{matrix}87&63\\54&21\end{matrix}$, $\alpha_1\alpha_1\alpha_2\beta_1\beta_1 \mid \alpha_2\beta_2\beta_2$,	$\begin{matrix}87&53\\64&21\end{matrix}$, $\alpha_1\alpha_1\beta_1 \mid \alpha_2\beta_1 \mid \alpha_2\beta_2\beta_2$,	$\begin{matrix}87&64\\52&31\end{matrix}$, $\alpha_1\alpha_1\alpha_2\beta_1 \mid \alpha_2\beta_2 \mid \beta_1\beta_2$,	$\begin{matrix}86&52\\73&41\end{matrix}$, $\alpha_1\beta_1 \mid \alpha_1\alpha_2\beta_2 \mid \beta_1 \mid \alpha_2\beta_2$,
$\begin{matrix}87&63\\52&41\end{matrix}$, $\alpha_1\alpha_1\alpha_2\beta_1\beta_2 \mid \alpha_2\beta_1\beta_2$,	$\begin{matrix}87&53\\62&41\end{matrix}$, $\alpha_1\alpha_1\beta_1 \mid \alpha_2\beta_2 \mid \alpha_2\beta_1\beta_2$,	$\begin{matrix}87&42\\65&31\end{matrix}$, $\alpha_1\alpha_1\beta_1\beta_1 \mid \alpha_2\beta_2 \mid \alpha_2\beta_2$,	$\begin{matrix}87&52\\63&41\end{matrix}$, $\alpha_1\alpha_1\beta_1 \mid \alpha_2\beta_2 \mid \beta_1 \mid \alpha_2\beta_2$,
$\begin{matrix}87&65\\42&31\end{matrix}$, $\alpha_1\alpha_1\alpha_2\alpha_2\beta_1\beta_2 \mid \beta_1\beta_2$,	$\begin{matrix}86&75\\42&31\end{matrix}$, $\alpha_1\alpha_2 \mid \alpha_1\alpha_2\beta_1\beta_2 \mid \beta_1\beta_2$,	$\begin{matrix}84&72\\63&51\end{matrix}$, $\alpha_1\alpha_2\beta_1\beta_2 \mid \alpha_1\beta_1 \mid \alpha_2\beta_2$,	$\begin{matrix}85&72\\63&41\end{matrix}$, $\alpha_1\alpha_2\beta_1 \mid \alpha_1\beta_2 \mid \beta_1 \mid \alpha_2\beta_2$,
$\begin{matrix}87&62\\54&31\end{matrix}$, $\alpha_1\alpha_1\alpha_2\beta_1\beta_1\beta_2 \mid \alpha_2\beta_2$,	$\begin{matrix}86&72\\54&31\end{matrix}$, $\alpha_1\alpha_2 \mid \alpha_1\beta_1\beta_1\beta_2 \mid \alpha_2\beta_2$,	$\begin{matrix}87&62\\53&41\end{matrix}$, $\alpha_1\alpha_1\alpha_2\beta_1\beta_2 \mid \beta_1 \mid \alpha_2\beta_2$,	$\begin{matrix}85&62\\73&41\end{matrix}$, $\alpha_1\beta_1 \mid \alpha_2 \mid \alpha_1\beta_2 \mid \beta_1 \mid \alpha_2\beta_2$.

A dividing line has been placed after each letter that has to be noted. Thence, by the rules given,

$$L_0(\infty\,;\,22\,;\,22) = 1,$$
$$L_1(\infty\,;\,22\,;\,22) = 2x^2+2x^3+3x^4+2x^5+2x^6,$$
$$L_2(\infty\,;\,22\,;\,22) = x^5+3x^6+4x^7+8x^8+4x^9+3x^{10}+x^{11},$$
$$L_3(\infty\,;\,22\,;\,22) = 2x^{10}+2x^{11}+3x^{12}+2x^{13}+2x^{14},$$
$$L_4(\infty\,;\,22\,;\,22) = x^{16},$$

supplying a complete verification of the work in Part II.

We have, therefore,

$$\text{GF}(l\,;\,22\,;\,22)$$
$$= L_0\frac{(l+1)\ldots(l+8)}{(1)\ldots(8)}+L_1\frac{(l)\ldots(l+7)}{(1)\ldots(8)}+L_2\frac{(l-1)\ldots(l+6)}{(1)\ldots(8)}$$
$$+L_3\frac{(l-2)\ldots(l+5)}{(1)\ldots(8)}+L_4\frac{(l-3)\ldots(l+4)}{(1)\ldots(8)}.$$

We have evidently, potentially, the complete solution of the problem of three-dimensional partition, and it remains to work it out and bring it to the same completeness as has been secured in this Part for the problem in two dimensions.

This will form the subject of Part VII. of this Memoir.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES